Belief Reasoning, Revision and Fusion by Matrix Algebra

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Abstract. Representation of belief states is an important issue for knowledge based systems. In this paper, we develop a matrix representation for ordered belief states and show that belief reasoning, revision and fusion can all be interpreted as operations of matrix algebra. Thus, the matrix representation can serve as the basis of algebraic semantics for belief logic.

Key Words: Belief states, matrix algebra, belief reasoning, belief revision, belief fusion, multi-agent systems.

1 Introduction

Knowledge representation is crucially important to knowledge-based applications. While we expect proper knowledge to be true, the term "knowledge" used in knowledge-based systems usually refers to information and belief in a broad sense. Therefore, knowledge can be fallible information in such applications. Consequently, the representation of belief states plays a key role in knowledge-based systems.

In many domains, such as epistemic reasoning[10, 11], belief revision[1, 14, 15] and belief fusion[2, 6, 16, 18–21, 23, 27], a knowledge base or a belief base consists of a set of sentences in a logical language. Since a belief base can be identified with the set of models satisfying it, an agent's belief state is usually represented as a subset of possible worlds.

However, some more finely tuned representations have been proposed, such as total pre-orders over the set of possible worlds [4, 7, 17, 24], ordinal conditional functions [5, 26, 28], possibility distributions[3, 8, 9], belief functions[25], and pedigreed belief states[12, 13]. One of the most popular representation formalisms is an ordering over the set of possible worlds.

A natural representation of an ordering can be achieved by using a Boolean matrix, if the set of possible worlds is finite. In this paper, we adopt such a representation and show that belief reasoning, revision and fusion can be interpreted in matrix algebra. Thus, the matrix representation can serve as the basis of algebraic semantics for belief logic.

2 Matrix Representation of Belief States

In this paper, A, B and M (possibly with subscripts) denote real-valued matrices. If M is a $n \times m$ matrix, then for $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$, $M(i,j)$, $M(i,\cdot)$ and $M(\cdot,j)$ denote the element in the (i,j) entry, the i-th row vector and the j-th column vector of the matrix respectively. Usually, we also write $M(i)$ and $M(j)$ for the row vector and the column vector if they are clear from the context. A *Boolean matrix* is a real-valued matrix whose elements all belong to the set $\{0,1\}$. Also, let I, $\mathbf{1}$ and $\mathbf{0}$ denote the unit square matrix, complete matrix and zero matrix respectively. In other words, I, $\mathbf{1}$ and $\mathbf{0}$ are Boolean matrices such that $I(i,j) = 1$ iff $i = j$, $\mathbf{1}(i,j) = 1$ and $\mathbf{0}(i,j) = 0$ for all i and j. The dimensions (i.e., numbers of rows and columns) of I, $\mathbf{1}$ and $\mathbf{0}$ depend on the context. If necessary, we will write $I_{n \times m}$, $\mathbf{1}_{n \times m}$, or $\mathbf{0}_{n \times m}$ to indicate their dimensions. Furthermore, we will identify a 1×1 matrix with its element.

The following operations of matrix algebra are extensively used in this paper.

Definition 1 *Let A and B be two matrices and r is a real number. The following binary operations can be defined when A and B have the same dimensions:*

1. *Max: $(A \vee B)(i,j) = \max(A(i,j), B(i,j))$,*
2. *Min: $(A \wedge B)(i,j) = \min(A(i,j), B(i,j))$,*
3. *Sum: $(A + B)(i,j) = A(i,j) + B(i,j)$,*
4. *Difference: $(A - B)(i,j) = A(i,j) - B(i,j)$,*
5. *Scalar product: $(r \cdot A)(i,j) = r \cdot A(i,j)$.*

When the column number of A is equal to the row number of B, another binary operation, the (Boolean) product, can be defined as:

$$(A \cdot B)(i,j) = \max_{0 \leq k \leq n-1} \min(A(i,k), B(k,j))$$

where n is the column number of A. Let $m \geq 0$, then A^m is inductively defined as $A^{m-1} \cdot A$ where $A^0 = I$. Furthermore, there are also several unary operations:

1. *Transposition: $A^t(i,j) = A(j,i)$,*
2. *Transitive closure: $A^+ = \bigvee_{m \geq 1} A^m$*
3. *Indicator: $A^d(i,j) = \begin{cases} 1, \text{ if } A(i,j) > 0 \text{ and } A(i,j) \geq A(j,i), \\ 0, \text{ otherwise.} \end{cases}$*

∎

If $\Phi_0 = \{p, q, \cdots\}$ is a finite set of atomic propositions and $\mathcal{L}$ is the propositional language based on Φ_0, then a possible world is a truth assignment $w : \Phi_0 \rightarrow \{0,1\}$. As usual, the domain of w can be extended to the set of all well-formed formulas (wffs) in $\mathcal{L}$. If $\mathcal{W} = \{w_0, w_1, \cdots, w_{n-1}\}$ denotes the set of all possible worlds, then a *belief state* is defined as an $n \times n$ Boolean matrix B. Intuitively, a belief state B defines an associated ordering relation $\prec_B$ over

$\mathcal{W}$ such that $B(i,j) = 1$ iff $w_j \prec_B w_i$. For each wff φ, the *characteristic matrix* of φ is the $n \times n$ Boolean matrix M_φ defined by

$$M_\varphi(i,j) = \begin{cases} 1, \text{ if } i = j \text{ and } w_i(\varphi) = 1, \\ 0, \text{ otherwise.} \end{cases}$$

Note that by this definition, we have the equations:

$$M_{\varphi \lor \psi} = M_\varphi \lor M_\psi$$

$$M_{\varphi \land \psi} = M_\varphi \land M_\psi$$

$$M_{\neg \varphi} = I - M_\varphi$$

where I is the $n \times n$ unit matrix.

3 Belief Reasoning, Revision and Fusion

Given the matrix representation of belief states and wffs, we can implement the tasks of belief reasoning, revision and fusion using matrix algebra.

3.1 Belief reasoning

Definition 2 *Given a belief state B and two wffs φ and ψ, we say that φ preferentially implies ψ under B, denoted by $\varphi \models_B \psi$, iff*

$$(\mathbf{1}_{1 \times n} - \mathbf{1}_{1 \times n} \cdot M_\psi) \cdot [\mathbf{1}_{n \times 1} - (M_\varphi \cdot B \cdot M_\varphi \lor M_{\neg \varphi}) \cdot \mathbf{1}_{n \times 1}] = 0,$$

where we assume $\cdot$ has precedence over other binary operations. ∎

The intuition of the above definition can be seen by interpreting a belief state as an ordering over possible worlds. Let $\prec$ be a binary relation over $\mathcal{W}$ and $X \subseteq \mathcal{W}$, then the *choice set*[12] of X with respect to $\prec$ is

$$C(X, \prec) = \{w \in X : \nexists w' \in X, w' \prec w\}.$$

Let the *truth set* of a wff φ be defined as $|\varphi| = \{w \in \mathcal{W} : w(\varphi) = 1\}$, then the set of minimal models of φ with respect to $\prec$, denoted by $Min(\varphi, \prec)$, is equal to $C(|\varphi|, \prec)$. The following proposition states the relationship between the preferential implication and minimal model belief reasoning.

Proposition 1 *Let B be a belief state and $\prec_B$ be its associated ordering, then for any wffs φ and ψ*

$$\varphi \models_B \psi \text{ iff } Min(\varphi, \prec_B) \subseteq |\psi|.$$

Proof: Let

$$A_1 = M_\varphi \cdot B \cdot M_\varphi \lor M_{\neg \varphi},$$

$$A_2 = \mathbf{1}_{n \times 1} - A_1 \cdot \mathbf{1}_{n \times 1},$$

$$A_3 = \mathbf{1}_{1 \times n} - \mathbf{1}_{1 \times n} \cdot M_\psi$$

then $A_1(i,j) = 1$ iff one of the following two conditions holds:

(i) $w_i \models \varphi$, $w_j \models \varphi$ and $w_j \prec_B w_i$, or

(ii) $i = j$ and $w_i \not\models \varphi$.

Thus, $\max_{0 \leq j \leq n-1} A(i,j) = 1$ iff $w_i \notin Min(\varphi, \prec_B)$. This results in $A_2(i,0) = 1$ iff $w_i \in Min(\varphi, \prec_B)$. On the other hand , $A_3(0,i) = 1$ iff $w_i \not\models \psi$. Therefore,

$$Min(\varphi, \prec_B) \subseteq |\psi|$$
$$\Leftrightarrow \not\exists 0 \leq i \leq n-1,\ w_i \in Min(\varphi, \prec_B) \text{ and } w_i \not\models \psi$$
$$\Leftrightarrow A_3 \cdot A_2 = \max_{0 \leq i \leq n-1} \min(A_3(1,i), A_2(i,1)) = 0$$
$$\Leftrightarrow \varphi \models_B \psi.$$

∎

Note that we do not impose any restriction on the matrix representation of belief states, so minimal models of a wff φ may not exist. Some reasonable restrictions have been proposed, such as transitivity and modularity[12]. However, our definition can still be used in the more restricted form of belief states.

3.2 Belief revision

Definition 3 *Let B be a belief state and φ be a wff, then the revised state of B by φ is defined as*

$$B^\circ \varphi = [B \vee (M_{\neg\varphi} \cdot \mathbf{1} \wedge \mathbf{1} \cdot M_\varphi)] \wedge (M_{\neg\varphi} \cdot \mathbf{1} \vee \mathbf{1} \cdot M_\varphi),$$

where the dimension of $\mathbf{1}$ is $n \times n$. ∎

This definition corresponds to a natural approach to revise an ordering over possible worlds with a wff. Let $\prec$ be an ordering relation over $\mathcal{W}$, then revising $\prec$ with a wff φ results in a new ordering $\prec' = Rev(\prec, \varphi)$ defined by

$$w \prec' w' \Leftrightarrow w(\varphi) = 1 \text{ and } w'(\varphi) = 0, \text{ or}$$
$$w(\varphi) = w'(\varphi) \text{ and } w \prec w'.$$

According to this definition, after the revision, all possible worlds satisfying φ are preferred over those that do not, whereas the ordering between other worlds is kept intact. The following proposition shows the equivalence between these two definitions.

Proposition 2 *Let B be a belief state and φ be a wff, then*

$$Rev(\prec_B, \varphi) = \prec_{B^\circ\varphi}$$

where $\prec_B$ and $\prec_{B^\circ\varphi}$ are, respectively, the associated ordering of B and $B^\circ\varphi$.

Proof: First, we note that for all $0 \leq i \leq n-1$, $(M_{\neg\varphi} \cdot \mathbf{1})(i,j) = 1$ iff $w_j(\varphi) = 0$ and for all $0 \leq j \leq n-1$, $(\mathbf{1} \cdot M_{\neg\varphi})(i,j) = 1$ iff $w_i(\varphi) = 1$. Let $A_1 = B \vee (M_{\neg\varphi} \cdot \mathbf{1} \wedge \mathbf{1} \cdot M_\varphi)$ and $A_2 = M_{\neg\varphi} \cdot \mathbf{1} \vee \mathbf{1} \cdot M_\varphi$, then

$$A_1(i,j) = 1 \Leftrightarrow B(i,j) = 1 \vee (w_i(\varphi) = 0 \wedge w_j(\varphi) = 1)$$

and

$$A_2(i,j) = 1 \Leftrightarrow w_i(\varphi) = 0 \vee w_j(\varphi) = 1$$
$$\Leftrightarrow w_i(\varphi) = w_j(\varphi) \vee (w_i(\varphi) = 0 \wedge w_j(\varphi) = 1).$$

Therefore,

$$B^\circ\varphi(i,j) = 1 \Leftrightarrow A_1(i,j) = 1 \wedge A_2(i,j) = 1$$
$$\Leftrightarrow (w_i(\varphi) = 0 \wedge w_j(\varphi) = 1) \vee (w_i(\varphi) = w_j(\varphi) \wedge B(i,j) = 1).$$

This is equivalent to

$$w_j \prec_{B^\circ\varphi} w_i \Leftrightarrow (w_i(\varphi) = 0 \wedge w_j(\varphi) = 1) \vee (w_i(\varphi) = w_j(\varphi) \wedge w_j \prec_B w_i),$$

so we have $\prec_{B^\circ\varphi} = Rev(\prec_B, \varphi)$.

■

An alternative revision strategy, called *natural revision*, has been proposed in [4]. In that operator, only minimal possible worlds satisfying φ in the original belief state are preferred over others. Consequently, more possible worlds remain intact in natural revision. Formally, a natural revision of $\prec$ with a wff φ results in a new ordering $\prec' = NRev(\prec, \varphi)$ defined by[1]

1. if $w \in Min(\varphi, \prec)$, then $w \prec' v$ for any $v \notin Min(\varphi, \prec)$ and
2. if $v, w \notin Min(\varphi, \prec)$, then $w \prec' v$ iff $w \prec v$.

The natural revision operator can be modelled in the matrix representation by the following definition.

Definition 4 *Let B be a belief state and φ be a wff. Define the column vector (i.e. $n \times 1$ matrices) $\mathbf{a}$ and the row vector(i.e. $1 \times n$ matrices) $\mathbf{b}$ as follows:*

$$\mathbf{a} = (M_\varphi \cdot B \cdot M_\varphi \vee M_{\neg\varphi}) \cdot \mathbf{1}_{n \times 1}$$

$$\mathbf{b} = (\mathbf{1}_{n \times 1} - \mathbf{a})^t.$$

Then, the naturally revised state of B by φ is defined as

$$B^*\varphi = [B \vee (\mathbf{1}_{n \times 1} \cdot \mathbf{b})] \wedge (\mathbf{a} \cdot \mathbf{1}_{1 \times n}).$$

■

Proposition 3 *Let B be a belief state and φ be a wff, then*

$$NRev(\prec_B, \varphi) = \prec_{B^*\varphi}$$

where $\prec_B$ and $\prec_{B*\alpha\varphi}$ are respectively the associated ordering of B and $B^*\varphi$.

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[1] In [4], natural revision operator is only applied to belief states represented by total pre-orders, so the definition here is a slight generalization of that given in [4].

Proof: Let $A = M_\varphi \cdot B \cdot M_\varphi \vee M_{\neg\varphi}$, then

$$A(i,j) = 1 \Leftrightarrow w_i(\varphi) = w_j(\varphi) = 1 \ \text{ and } \ w_j \prec_B w_i, \ \text{ or}$$
$$i = j \ \text{ and } \ w_i(\varphi) = 0.$$

Thus, $\mathbf{a}(i,0) = \max_{0 \le j \le n-1} A(i,j) = 1$ iff $w_i \notin Min(\varphi, \prec_B)$ and $\mathbf{b}(0,j) = 1$ iff $w_j \in Min(\varphi, \prec_B)$. Therefore, $B^*\varphi(i,j) = 1$ iff the following two conditions are simultaneously satisfied:

1. $w_j \prec_B w_i$ or $w_j \in Min(\varphi, \prec_B)$;
2. $w_i \notin Min(\varphi, \prec_B)$.

Consequently, we have

$$w_j \prec_{B^*\varphi} w_i \Leftrightarrow w_j \in Min(\varphi, \prec_B) \ \text{ and } \ w_i \notin Min(\varphi, \prec_B) \ \text{ or}$$
$$w_j \prec_B w_i \ \text{ and } \ w_i, w_j \notin Min(\varphi, \prec_B),$$

so $NRev(\prec_B, \varphi) = \prec_{B^*\varphi}$ by the definition.

∎

3.3 Belief fusion

In belief fusion, the notion of pedigreed belief states proposed in $[12, 13]$ is a very general one. The advantage of the pedigreed belief state representation is that it can keep track of information about the credibility of sources. By the matrix notation, we can easily represent a pedigreed belief state and the belief state induced from it. Let Ω be a set of information sources, then it is assumed that for each source $s \in \Omega$, there is a belief state B_s associated with it. It is also assumed that the credibility of information sources is determined by a ranking function $rank : \Omega \to \mathcal{N}$, where $\mathcal{N}$ is the set of positive integers.

Definition 5 *Let Ω be a set of information sources and $rank : \Omega \to \mathcal{N}$ be its ranking function, then*

1. *the pedigreed belief state of qualitatively fusing Ω is*

$$PB_q(\Omega) = \bigvee_{s \in \Omega} rank(s) \cdot B_s,$$

2. *the pedigreed belief state of arithmetically fusing Ω is*

$$PB_a(\Omega) = \sum_{s \in \Omega} rank(s) \cdot B_s,$$

3. *the pedigreed belief state of exponentially fusing Ω is*

$$PB_e(\Omega) = \sum_{s \in \Omega} \alpha^{rank(s)-1} \cdot B_s,$$

where $\alpha \ge |\Omega|$ is a real number,

4. the belief state of fusing Ω (qualitatively, arithmetically, or exponentially) is

$$\oplus_x(\Omega) = (PB_x(\Omega))^d$$

where $x = q, a$, or e. ∎

The definition of qualitative fusion is taken from [12]. In that work, a *generalized belief state* is defined as a modular and transitive relation over $\mathcal{W}$. A relation $\prec$ over $\mathcal{W}$ is transitive if $w_1 \prec w_2$ and $w_2 \prec w_3$ imply $w_1 \prec w_3$ and modular if $w_1 \prec w_2$ implies that for any $w \in \mathcal{W}$, $w \prec w_2$ or $w_1 \prec w$. Therefore, each information source s is associated with a generalized belief state $\prec^s$ and an aggregation operator $AGRRf$ is defined for a set of information sources S as

$$AGRRf(S) = \{(w,v) \mid \exists s \in S.\ w \prec^s v \wedge (\forall s' \sqsupset s \in S.\ w \approx^{s'} v)\}$$

where $s' \sqsupset s$ means that $rank(s') > rank(s)$ and $w \approx^{s'} v$ means that $\neg(w \prec^{s'} v) \wedge \neg(v \prec^{s'} w)$. The pedigreed belief state for a set of information sources S is then defined as a pair $(\prec, l)$, where $\prec = AGRRf(S)$ and $l : \prec \to \mathcal{N}$ such that

$$l((w,v)) = \max\{rank(s) \mid w \prec^s v, s \in S\}.$$

Note that in our definition, each information source is associated with a belief state in matrix form, whereas in [12], it is associated with a generalized belief state in the form of a modular and transitive relation. Let Ω denote a set of information sources in our definition such that each $\prec_{B_s}$ is transitive and modular. Then, we can define S_Ω as the corresponding set of information sources in the sense of [12], so that each $s \in S_\Omega$ is associated with the ordering $\prec_{B_s}$.

Proposition 4 *Let Ω be a set of information sources in our definition, then*

$$AGRRf(S_\Omega) = \prec_{\oplus_q(\Omega)}$$

and for each $0 \le i, j \le n - 1$ such that $\oplus_q(\Omega)(i,j) = 1$,

$$l((w_j, w_i)) = PB_q(\Omega)(i,j).$$

Proof:

1. On the one hand , using the definition of indicator operation, for all $0 \le i, j \le n - 1$, $w_j \prec_{\oplus_q(\Omega)} w_j$ iff $PB_q(\Omega)(i,j) > 0$ and $PB_q(\Omega)(i,j) \ge PB_q(\Omega)(j,i)$. Since $PB_q(\Omega)(i,j) = \max_{s \in \Omega} rank(s) \cdot B_s(i,j)$, $PB_q(\Omega)(i,j) > 0$ implies $\exists s \in S$, $B_s(i,j) = 1$, i.e. $\exists s \in S, w_j \prec_{B_s} w_i$. Let s_0 denote such an information source such that $rank(s_0) = PB_q(\Omega)(i,j)$, then $PB_q(\Omega)(i,j) \ge PB_q(\Omega)(j,i)$ implies $\forall s \sqsupset s_0$, $B_s(i,j) = B_s(j,i) = 0$, i.e. $\forall s \sqsupset s_0$, $w_j \approx^s w_i$. Therefore, $w_j \prec_{\oplus_q(\Omega)} w_j$ implies $(w_j, w_i) \in AGRRf(S_\Omega)$. On the other hand , if $(w_j, w_i) \in AGRRf(S_\Omega)$, then there exists $s \in \Omega$ such that $B_s(i,j) = 1$ and for all s' such that $rank(s') > rank(s)$, $B_{s'}(j,i) = B_{s'}(i,j) = 0$. Therefore, $PB_q(\Omega)(i,j) > 0$ and $PB_q(\Omega)(i,j) \ge PB_q(\Omega)(j,i)$ hold. This in turn implies $w_j \prec_{\oplus_q(\Omega)} w_j$.

2. $PB_q(\Omega)(i,j) = \max_{s \in \Omega} rank(s) \cdot B_s(i,j) = \max\{rank(s) \mid w_j \prec_{B_s} w_i, s \in \Omega\} = l((w_j, w_i))$

■

While the operator $AGRRf(S)$ preserves the modularity of the generalized belief states associated with sources in S, it does not preserve the transitivity. The resultant belief state for the fusion is, in fact, the transitive closure of $AGRRf(S)$. Therefore, the definition of qualitative fusion is for a very general notion of belief states according to the matrix representation. If some additional properties (such as modularity and transitivity) are imposed on the definition of belief states, some further operations (such as transitive closure) are required for implementation of the belief fusion operator.

In [12], it is suggested that aggregation of sources based on a general framework of voting is also possible. While qualitative fusion accounts for the ranking of sources supporting or disagreeing with an opinion, it does not count the percentage of sources in each camp. The definition of arithmetic fusion implements the idea in a straightforward way by algebraic operations. When all sources are equally credible, this corresponds to merging by majority in the spirit of [22].

However, if the credibility of the sources are highly variant, it is possible that many sources of low credibility will outweigh one of high credibility. To circumvent this problem, we can use the exponential fusion. Since the base of the exponential terms is not less than the number of sources, it is guaranteed that the opinion of higher ranks will dominate all opinions of lower ranks.

4 Conclusion

In this paper, we show that matrix algebra can be applied to belief reasoning, revision and fusion in knowledge-based systems. By employing a Boolean matrix representation of belief states, the main tasks for belief or knowledge management in such systems can be reduced to algebraic operations in matrix algebra. This enables a uniform treatment of these reasoning tasks.

To make our approach general, we do not impose any restrictions on the matrix representation of belief states. However, in real applications, belief states are usually assumed to have some special structures, such as modularity and transitivity. In such cases, the closure of an algebraic operation with respect to these additional properties will become very important. In the further work, we will investigate how the closure property may influence the representation of belief states and the choice of algebraic operations.

Furthermore, the dimensions of matrices used in the representation of belief states are, in general, exponential with respect to the number of propositional variables. Since the dimensions of these matrices may be very large, it is impractical to implement reasoning tasks of belief logic directly via the semantic representation. Instead, a proof theory for belief logic should be developed and the matrix representation proposed in this paper can serve as the basis of an algebraic semantics for such logic.

Acknowledgements

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