Exponential Stabilization for Caplygin Systems Based on A Simplified Rank Condition

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Abstract

The paper investigates the global ρ-exponential stabilizability of nonholonomic Caplygin systems. A novel decomposition of state is given first. When systems are linear in certain state variables, a simple and easily verified rank condition can be proposed to guarantee the global ρ-exponential stabilizability of Caplygin systems. In our design, all parameters can be explicitly determined from the constraint function. Moreover, an interesting coordinate transformation can be used to change a Caplygin system into another one so that the proposed criterion can be applied to various situations. For an important class of Caplygin systems, the rank condition is further reduced to some conditions relating to the degree and non-zero property of the lowest polynomials of constraint function. Several interesting examples including of the knife-edge, the extended power-form, the rolling wheel and hopping robot systems are shown that they can be exponentially stabilized by an easy test.

Keywords: **ρ-exponential stabilizability, Caplygin systems, rank condition, decomposition, coordinate transformation.**

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I. Introduction

The paper investigates the exponential stabilizability of Caplygin systems that can be described as follows:

$$\ddot{q}_1 = u \tag{1}$$

$$\dot{q}_2 = - J(q_1)\dot{q}_1, \tag{2}$$

where $q_1 \in \Re^n$, $q_2 \in \Re^m$ and $u \in \Re^n$; the constrain function J is a matrix-valued analytic function defined on $\Re^n$ [2]. The target is to provide a simple and easily checked criterion for the exponential stabilizability of Caplygin systems.

Caplygin systems as a subclass of nonholonomic systems were introduced in [2] for the control community. Practical examples includes the knife edge, the two-wheel mobile robots, the rolling wheel, and extended power form, e.t.c., [2], [6]. In recent years, the interests for such systems were from the fact that they have no static time-invariant continuous stabilizers [3]. Simultaneously, there are no time-varying smooth controllers that can exponentially stabilize them. To overcome this drawback, several approaches, such as the homogeneous and the discontinuous feedback methods, were proposed to guarantee the exponential stabilizability in present literature [1], [5], [7]-[8], [10]. Observe these results, the homogeneous feedback method needs some special construction for the controllers and thus can be applied only to certain specific systems. In the contrast, the discontinuous feedback method can be modified to solve the exponential stabilizability problem for a large class of systems as shown in [7].

This paper will adopt the latter approach and propose a further improvement for Caplygin systems based on the results given in [7]. Indeed, a novel decomposition will be propose such that a Caplygin system can be transformed into a cascade system. Then, a simplified controllability-like rank condition will be proposed by borrowing the main result in that paper. Comparing with [7], all parameters can be explicitly determined from the constraint function J in this paper. Moreover, an interesting coordinate transformation can be used to change a Caplygin system into another one so that the proposed criterion can be applied to various situations. For an important class of Caplygin systems, the rank condition will be further reduced to certain easily tested conditions relating to the degree and non-zero property of the lowest polynomials of constraint function J. Several interesting examples including of the knife-edge, the extended power-form and the rolling wheel systems will be given to demonstrate that they satisfy the proposed conditions by an easy test and thus can be exponentially stabilized. As a final example, a set-point control problem for the hopping robot system is also studied and shown that an exponential convergence result can be attained based on our approaches. From these applications, it can be seen that the proposed criterion does provide a direct and simple test with respect to the results given in present literature for determining whether a Caplygin system can be exponentially stabilized.

II. Brief Review of Newly Developed Criterion

In this section, an exponential stabilizability criterion given in [7] will be reviewed briefly. Firstly, several basic definitions are recalled. Throughout this paper, let $\Re^{n\times m}$ denote the set of all $n\times m$ matrices and D_r denote the diagonal matrix with diagonal elements taken from the elements of a vector r. Let $A:\Re^{n_1}\times\Re^{m_1}\to\Re^{n_2\times m_2}$ be a matrix-valued function. Suppose the elements of A are all analytic.

Definition 1. For any $1\le i\le n_2$ and $1\le j\le m_2$, let $p_{ij}^A(v,w)$ denote the lowest homogeneous polynomial in the Taylor expansion of the (i,j) entry of A at the origin and d_{ij}^A denote the degree of p_{ij}^A. Notice that, $p_{ij}^A\equiv 0$ when the (i,j) entry of A is the zero function. In this case, we define $d_{ij}^A=\infty$.

Definition 2. Let $(r,s)\in\aleph^{n_2}\times\aleph^{m_2}$ be any integer-valued vector satisfying $r_i\le s_j+d_{ij}^A,\ \forall i,\forall j$. Let $\bar{p}_{ij}^{rsA}(v,w)$ be defined as follows:

$$\bar{p}_{ij}^{rsA}=\begin{cases}p_{ij}^A, & \text{if } r_i=s_j+d_{ij}^A,\\ 0, & \text{if } r_i<s_j+d_{ij}^A.\end{cases}\tag{3}$$

In the following, let us recall the definitions of dilation operation, homogeneous norm and global ρ-exponential stability given in [7] and [8].

Definition 3. Let $v=(v_1,v_2,\cdots,v_{n_2})^T\in\Re^{n_2}$. A dilation $\Delta_\zeta^r:\Re^{n_2}\to\Re^{n_2}$ on $\Re^{n_2}$ is defined by assigning n_2 real numbers $r=(r_1,r_2,\cdots,r_{n_2})$ and a nonzero real number ζ such that $\Delta_\zeta^r v=(\zeta^{r_1}v_1,\zeta^{r_2}v_2,\cdots,\zeta^{r_{n_2}}v_{n_2})$. Similarly, let $A=(a_{ij})\in\Re^{n_2\times m_2}$. A dilation $\Delta_\zeta^{rs}:\Re^{n_2\times m_2}\to\Re^{n_2\times m_2}$ on $\Re^{n_2\times m_2}$ is defined by assigning n_2+m_2 real numbers $r=(r_1,r_2,\cdots,r_{n_2})$ and $s=(s_1,s_2,\cdots,s_{m_2})$, and a nonzero real number ζ such that $\Delta_\zeta^{rs}A=(\zeta^{r_i-s_j}a_{ij})$.

Definition 4. A positive definite continuous function $\rho:\Re^{\hat{n}}\to\Re$ is called a homogeneous norm w. r. t. the dilation Δ_ζ^r if $\rho(\Delta_\zeta^r x)=\zeta\rho(x),\forall\zeta\neq0,\forall x\in\Re^{\hat{n}}$.

Definition 5. The equilibrium point $x=0$ is globally ρ-exponentially stable if there exist a homogeneous norm ρ and two positive constants c_1 and c_2 such that for any solution $x(t)$, the following inequality holds:

$$\rho(x(t))\le\sigma_1\rho(x(t_0))e^{-\sigma_2(t-t_0)},\quad\forall t\ge t_0. \tag{4}$$

In the remainder of this section, let us consider a class of cascaded systems described in the following form

$$\dot{x}_1=A_1x_1+B_1u_1 \tag{5}$$

$$\dot{x}_2=A_2(x_1,u_1)\,x_2+B_2(x_1,u_1)\,u_2, \tag{6}$$

where $x_i\in\Re^{n_i}$ and $u_i\in\Re^{m_i}$, $\forall\,i=1,2$; A_1 and B_1 are matrices with suitable dimensions; $A_2(x_1,u_1)$ and $B_2(x_1,u_1)$ are analytic matrix-valued functions. Assume that the following hypothesis holds.

(H1) There exists an integer-valued vector $(r,s)\in\aleph^{n_2}\times\aleph^{m_2}$ satisfying the following inequalities

$$r_i \le r_j + d_{ij}^{A_2} \ \text{ and } \ r_i \le s_{\tilde{j}} + d_{i\tilde{j}}^{B_2}, \forall i, \forall j, \forall \tilde{j}. \tag{7}$$

Under (H1), $\bar{p}_{ij}^{rrA_2}(x_1,u_1)$ and $\bar{p}_{i\tilde{j}}^{rsB_2}(x_1,u_1)$ can be defined as (3). For any positive constant k, denote $\bar{A}_2(x_1,u_1) = kD_r + (\bar{p}_{ij}^{rrA_2}(x_1,u_1))$ and $\bar{B}_2(x_1,u_1) = (\bar{p}_{i\tilde{j}}^{rsB_2}(x_1,u_1))$. Then, the following result can be proposed. It was proven in [7] and can be viewed as a preliminary result in our study of the exponential stabilizability for Caplygin systems.

Proposition 1. Consider a system of the form (5)-(6). Suppose that (H1) and the following hypothesis hold for some integer-valued vector $(r,s) \in \aleph^{n_2} \times \aleph^{m_2}$.

(H2) For some positive constant k and some vector $(a,b) \in \Re^{n_1} \times \Re^{m_1}$ satisfying $(kI + A_1)a + B_1 b = 0$, the pair (A_1, B_1) and $(\bar{A}_2(a,b), \bar{B}_2(a,b))$ are both controllable.

Let $\hat{r} = (\underbrace{1,\ldots,1}_{n_1}, r) \in \Re^{n_1+n_2}$ and $K_1 \in \Re^{m_1 \times n_1}$ and $K_2 \in \Re^{m_2 \times n_2}$ be two matrices such that the matrices $kI + A_1 + B_1 K_1$ and $\bar{A}_2(a,b) + \bar{B}_2(a,b) K_2$ are both stable. Then, the origin of the closed-loop system is globally ρ-exponentially stable when the controller (u_1,u_2) is chosen as in the following

$$u_1 = (b - K_1 a)\lambda + K_1 x_1 \ \ , \ \ u_2 = \begin{cases} \Delta_\lambda^{sr} K_2 x_2, & if \ (x_1(t_0), x_2(t_0)) \neq 0 \\ 0, & if \ (x_1(t_0), x_2(t_0)) = 0, \end{cases} \tag{8}$$

for any $t_0 \ge 0$ where $\lambda = e^{-k(t-t_0)}\rho(x_1(t_0), x_2(t_0))$ with ρ being any homogeneous norm w.r.t. dilation $\Delta_\zeta^{\hat{r}}$. ■

III. ρ-Exponential Stabilizability via A Simplified Rank Condition

A. Rank condition and ρ-exponential stabilizability

In this subsection, a simplified rank condition will be given to achieve the ρ-exponential stabilizability for Caplygin systems based on Proposition1. To this end, the following assumption is useful to transform a Caplygin system into a system of the form (5)-(6).

(C1) (Linear in partial state variables). There exist a decomposition of partial state vector q_1 and input vector u defined as $q_1 = [z_1{}^T z_2{}^T]^T$ and $u = [u_1^T \quad u_2^T]^T$ with $z_1 \in \Re^{\bar{n}_1}$, $z_2 = (z_{21}, z_{22}, \cdots, z_{2\bar{n}_2}) \in \Re^{\bar{n}_2}, u_1 \in \Re^{\bar{n}_1}, u_2 \in \Re^{\bar{n}_2}$ and $n = \bar{n}_1 + \bar{n}_2$ so that the constraint function $J(z_1, z_2) = [\sum_{j=1}^{\bar{n}_2} J_{1j}(z_1) z_{2j} \quad \bar{J}_2(z_1)]$ with $J_{1j} : \Re^{\bar{n}_1} \to \Re^{m \times \bar{n}_1}$, $\forall 1 \le j \le \bar{n}_2$, and $\bar{J}_2 : \Re^{\bar{n}_1} \to \Re^{m \times \bar{n}_2}$ being matrix-valued analytic functions (only depending on the state variables z_1).

For the compactness, we define a matrix-valued function $\bar{J}_1 : \Re^{\bar{n}_1} \times \Re^{\bar{n}_1} \to \Re^{m \times \bar{n}_2}$ as

$$\bar{J}_1(z_1, z_3) = [J_{11}(z_1) z_3 \quad J_{12}(z_1) z_3 \quad \cdots \quad J_{1\bar{n}_2}(z_1) z_3], \quad \forall z_1 \in \Re^{\bar{n}_1}, \forall z_3 \in \Re^{\bar{n}_1}. \tag{9}$$

Then, it is straightforward to see that $J(q_1)\dot{q}_1 = \bar{J}_1(z_1, \dot{z}_1) z_2 + \bar{J}_2(z_1) \dot{z}_2$ and

$$\bar{J}_1(z_1, \zeta z_3) = \zeta \bar{J}_1(z_1, z_3), \forall z_1, z_3 \in \Re^{n_1}, \forall \zeta \in \Re. \tag{10}$$

Let $x_1 = [z_1^T \ \dot{z}_1^T]^T \in \Re^{n_1}$ and $x_2 = [q_2^T \ z_2^T \ \dot{z}_2^T]^T \in \Re^{n_2}$ with $n_1 = 2\bar{n}_1$ and $n_2 = m + 2\bar{n}_2$. Then, a Caplygin system of the form (1)-(2) can be rewritten into the form of (5)-(6) where the matrices A_1 and B_1, and the matrix-valued functions A_2 and B_2 can be described as follows:

$$A_1 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\bar{J}_1 & -\bar{J}_2 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}. \tag{11}$$

It is easy to see that (A_1, B_1) is in the controllable canonical form (CCF) [4]. To verify (H1), let us define the following parameters

$$\bar{d}_i = \min_{1 \le j_1 \le \bar{n}_2, 1 \le j_2 \le \bar{n}_2} (d_{ij_1}^{\bar{J}_1}, d_{ij_2}^{\bar{J}_2}) \text{ and } \bar{r}_i = \bar{d}_i + 1, \quad \forall 1 \le i \le m. \tag{12}$$

By the direct computation, it can be checked that the "degree matrices" of A_2 and B_2 can be described as follows

$$(d_{ij}^{A_2}) = \begin{bmatrix} \infty & (d_{ij_1}^{\bar{J}_1}) & (d_{ij_2}^{\bar{J}_2}) \\ \infty & \infty & 0 \\ \infty & \infty & \infty \end{bmatrix} \quad \text{and} \quad (d_{i\tilde{j}}^{B_2}) = \begin{bmatrix} \infty \\ \infty \\ 0 \end{bmatrix}. \tag{13}$$

Let $\bar{d} = (\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_m)$, $\bar{r} = (\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_m)$, $r = (\bar{r}, \underbrace{1, \ldots, 1}_{2\bar{n}_2})$, $s = (\underbrace{1, 1, \cdots, 1}_{\bar{n}_2})$ and a matrix

$E_{\bar{d}} = [\underbrace{\bar{d}^T \ \bar{d}^T \cdots \bar{d}^T}_{\bar{n}_2}] \in \Re^{m\times\bar{n}_2}$. Then, the following inequalities hold:

$$(r_i - r_j) = \begin{bmatrix} * & \mathrm{E}_{\bar{d}} & \mathrm{E}_{\bar{d}} \\ -\mathrm{E}_{\bar{d}}^T & 0 & 0 \\ -\mathrm{E}_{\bar{d}}^T & 0 & 0 \end{bmatrix} \le (d_{ij}^{A_2}) \ \text{ and } \ (r_i - s_{\tilde{j}}) = \begin{bmatrix} E_{\bar{d}} \\ 0 \\ 0 \end{bmatrix} \le (d_{i\tilde{j}}^{B_2}). \tag{14}$$

Now, (H1) follows from the inequalities above.

In the following, let us compute the matrix-valued functions $\bar{A}_2$ and $\bar{B}_2$. First, for each $1 \le i \le m, 1 \le j_1 \le \bar{n}_2, 1 \le j_2 \le \bar{n}_2$, define the homogeneous polynomials $\bar{p}_{ij_1}^1 : \Re^{\bar{n}_1} \times \Re^{\bar{n}_1} \to \Re$ and $\bar{p}_{ij_2}^2 : \Re^{\bar{n}_1} \to \Re$ relating to $\bar{J}_1$ and $\bar{J}_2$, respectively, as follows:

$$\bar{p}_{ij_1}^1 = \begin{cases} p_{ij_1}^{\bar{J}_1}, \mathit{if}\ d_{ij_1}^{\bar{J}_1} = \bar{d}_i, \\ 0, \quad \mathit{otherwise}, \end{cases} \quad \text{and} \quad \bar{p}_{ij_2}^2 = \begin{cases} p_{ij_2}^{\bar{J}_2}, \mathit{if}\ d_{ij_2}^{\bar{J}_2} = \bar{d}_i, \\ 0, \quad \mathit{otherwise.} \end{cases} \tag{15}$$

Denote $\bar{P}_1(z_1) = (\bar{p}_{ij_1}^1(z_1, z_1))$ and $\bar{P}_2(z_1) = (\bar{p}_{ij_2}^2(z_1))$ for all z_1 in $\Re^{\bar{n}_1}$. Notice that for any positive constant k, every solutions $(a,b) \in \Re^{2\bar{n}_1} \times \Re^{\bar{n}_1}$ satisfying the equation $(kI - A_1)a + B_1 b = 0$ can be described as $a = [\eta^T, -k\eta^T]^T$ and $b = k^2\eta$ for all $\eta \in \Re^{\bar{n}_1}$ in view of the form of A_1 and B_1 given in (11). Then, we have

$$(\bar{p}_{ij_1}^1(\eta^T, -k\eta^T)) = -k(\bar{p}_{ij_1}^1(\eta^T, \eta^T)) = -k\bar{P}_1(\eta), \forall \eta \in \Re^{\bar{n}_1}, \tag{16}$$

by employing the definitions of $\bar{p}_{ij_1}^1$ above and the property of $\bar{J}_1$ given in (10). Since $\bar{J}_1$ and $\bar{J}_2$ only depends on $x_1 = (z_1, \dot{z}_1)$ (and independent on u_1), the matrix-valued function $\bar{A}_2(x_1, u_1) \equiv \bar{A}_2(x_1)$ is also independent on u_1. Thus, the function $\bar{p}_{ij}^{rrA_2}$ described in Definition 2 can be viewed as a function defined on $\Re^{\bar{n}_1} \times \Re^{\bar{n}_1}$. Moreover, it can be checked that

$$\bar{p}_{ij}^{rrA_2} = \begin{cases} \bar{p}_{ij}^1, & \mathit{if}\ 1 \le i \le m,\ m+1 \le j \le m + \bar{n}_2, \\ \bar{p}_{ij}^2, & \mathit{if}\ 1 \le i \le m,\ m + \bar{n}_2 + 1 \le j \le m + 2\bar{n}_2, \\ \delta_{(i-m)(j-m-\bar{n}_2)}, & \mathit{if}\ m+1 \le i \le m + \bar{n}_2,\ m + \bar{n}_2 + 1 \le j \le m + 2\bar{n}_2, \\ 0, & \mathit{otherwise}, \end{cases} \tag{17}$$

in view of (11)-(14) and by the definition of $\bar{p}_{ij}^{rrA_2}$ where $(\delta_{ij}) = I$ is the $\bar{n}_2 \times \bar{n}_2$ identity matrix.

Thus, the matrix-valued functions $\overline{A}_2$ can be explicitly written as

$$\overline{A}_2(a) = kD_r + (\overline{p}_{ij}^{rrA_2}(\eta,-k\eta)) = \begin{bmatrix} k\ D_{\bar{r}} & -(\overline{p}_{ij}^1(\eta,-k\eta)) & -(\overline{p}_{ij}^2(\eta)) \\ 0 & kI & I \\ 0 & 0 & kI \end{bmatrix} = \begin{bmatrix} k\ D_{\bar{r}} & k\overline{P}_1(\eta) & -\overline{P}_2(\eta) \\ 0 & kI & I \\ 0 & 0 & kI \end{bmatrix}, \quad (18)$$

for all $k > 0$, all $\eta \in \Re^{\bar{n}_1}$ with $a = [\eta^T, -k\eta^T]^T$. Similarly, it can be verified that $\overline{B}_2 = (\overline{p}_{\bar{i}\bar{j}}^{rsB_2}) \equiv B_2 = \begin{bmatrix} 0 & 0 & I \end{bmatrix}^T$. To check the controllability of the pair $(\overline{A}_2(a), \overline{B}_2)$, let us assume that the following condition holds.

(C2) (Reduced order controllability). Suppose $rank[\overline{P}(\eta_0), D_{\bar{r}}\overline{P}(\eta_0), \cdots, D_{\bar{r}}^{m-1}\overline{P}(\eta_0)] = m$ for some $\eta_0 \in \Re^{\bar{n}_1}$, where

$$\overline{P}(\eta) = \overline{P}_1(\eta) - (D_{\bar{d}})\overline{P}_2(\eta). \quad (19)$$

Before verify the controllability of $(\overline{A}_2(a), \overline{B}_2)$, we need a technique lemma stated as follows.

Lemma 1. Consider three matrices $A \in \Re^{m \times \bar{n}}$, $B \in \Re^{m \times p}$ and $C \in \Re^{\bar{n} \times \bar{n}}$. Suppose the matrix C is invertible and $rank(B) = m$. Then, the following equality holds:

$$rank\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = m + \bar{n}.$$

Proof. It only needs to show that for any vector $(y_1, y_2) \in \Re^m \times \Re^{\bar{n}}$, there exists a vector $(\zeta_1, \zeta_2) \in \Re^{\bar{n}} \times \Re^p$ such that the following equality holds:

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

From the equation above, it is sufficient to choose $\zeta_1 = C^{-1}y_2$. Since $rank(B) = m$, there exists also a vector $\zeta_2 \in \Re^p$ such that $B\zeta_2 = y_1 - A\zeta_1$. It is straightforward to see that the equation described in the above holds by the choice of (ζ_1, ζ_2). The proof of the lemma is completed. ■

Under condition (C2), the controllability of $(\overline{A}_2(a), \overline{B}_2)$ can be guaranteed in the following result.

Proposition 2. Suppose (C2) holds for some $\eta_0 \in \Re^{\bar{n}_1}$. Let k be any given positive constant,

$\overline{B}_2 = [0 \quad 0 \quad I]^T$ and the matrix-valued function $\overline{A}_2(a)$ be defined as in (18) with $a = [\eta_0{}^T \quad -k\eta_0{}^T]^T$. Then, $(\overline{A}_2(a), \overline{B}_2)$ is controllable.

Proof. First, notice that $D_{\bar{r}}$ - $I = D_{\bar{d}}$ in view of (12). According to (18) and by employing the equation above, the following equality relating to the controllability matrix can be derived via elementary column operations:

$$rank[\overline{B}_2, \overline{A}_2(a)\overline{B}_2, \overline{A}_2^2(a)\overline{B}_2, \cdots, \overline{A}_2^{m+1}(a)\overline{B}_2] = rank[\overline{B}_2, \overline{B}_3, \overline{B}_4, \overline{B}_5, \cdots, \overline{B}_{m+3}], \tag{20}$$

where $\overline{B}_3 = \overline{A}_2(a)\overline{B}_2 - k\overline{B}_2 = [-\overline{P}_2^T(\eta_0)\, I \quad 0]^T$, $\overline{B}_4 = (1/k)(\overline{A}_2(a)\overline{B}_3 - k\overline{B}_3) = [\overline{P}^T(\eta_0) \quad 0 \quad 0]^T$ and $\overline{B}_{i+4} = (1/k)(\overline{A}_2(a_0)\overline{B}_{i+3}) = \begin{bmatrix} D_r^i \overline{P}(\eta_0) \\ 0 \\ 0 \end{bmatrix}$, $\forall 1 \le i \le m-1$. Since $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is invertible, we have

$$rank[\overline{B}_2, \overline{B}_3, \overline{B}_4, \overline{B}_5, \cdots, \overline{B}_{m+3}] = rank \begin{bmatrix} 0 & -\overline{P}_2(\eta_0) & \overline{P}(\eta_0) & D_r\overline{P}(\eta_0) & \cdots & D_r^{m-1}\overline{P}(\eta_0) \\ 0 & I & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = m + 2\overline{n}_2,$$

in view of (C2) and Lemma 1. This results in the controllability of $(\overline{A}_2(a), \overline{B}_2)$ according to (20). ■

Remark 1. In fact, it is not difficult to show that (C2) is also a necessary condition of the controllability of $(\overline{A}_2(a), \overline{B}_2)$. Since we don't need this property, its proof is omitted. ■

The following theorem is readable from Propositions 1-2 and the previous discussions.

Theorem 1. Consider a Caplygin system of the form (1)-(2). Suppose (C1)-(C2) hold for some $\eta_0 \in \Re^{\overline{n}_1}$. Let $\overline{d}_i$ and $\overline{r}_i$ be the constants defined in (12) for each $1 \le i \le m$. For any positive constant k, let $K_1 = [K_{11}\ K_{12}] \in \Re^{n_1 \times n_1} \times \Re^{n_1 \times n_1}$ and $K_2 = [K_{21}\ K_{22}\ K_{23}] \in \Re^{n_2 \times m} \times \Re^{n_2 \times n_2} \times \Re^{n_2 \times n_2}$ be two matrices such that the following matrices

$$\begin{bmatrix} kI & I \\ K_{11} & kI+K_{12} \end{bmatrix} \text{ and } \begin{bmatrix} kD_{\bar{r}} & k\overline{P}_1(\eta_0) & -\overline{P}_2(\eta_0) \\ 0 & kI & I \\ K_{21} & K_{22} & kI+K_{23} \end{bmatrix} \tag{21}$$

are both stable. Choose the controller (u_1, u_2) as in the following

$$u_1 = (k^2 I - K_{11} + kK_{12})\eta_0 \lambda + K_{11} z_1 + K_{12} \dot{z}_1 \tag{22}$$

and

$$u_2 = \begin{cases} K_{21}(\Delta_{1/\lambda}^{\bar{d}} q_2) + K_{22} z_2 + K_{23} \dot{z}_2, \; if \; (q_1(t_0), \dot{q}_1(t_0), q_2(t_0)) \neq 0, \\ 0, \qquad if \, (q_1(t_0), \dot{q}_1(t_0), q_2(t_0)) = 0. \end{cases} \tag{23}$$

where $\lambda = e^{-k(t-t_0)} \rho(q_1(t_0), \dot{q}_1(t_0), q_2(t_0))$ with ρ being any homogeneous norm w.r.t. dilation vector $\tilde{r} = (\underbrace{1,1,\cdots,1}_{2n}, \bar{r})$. Then, the origin of the closed-loop system is globally ρ-exponentially stable.

Proof. Based on previous discussions and Proposition 2, hypotheses (H1)-(H2) hold for $r = (\bar{r}, \underbrace{1,\ldots,1}_{2\bar{n}_2})$, $s = (\underbrace{1,1,\cdots,1}_{\bar{n}_2})$, $a = [\eta_0{}^T \; -k\eta_0{}^T]^T$ and $b = k^2 \eta_0$. Moreover, the homogeneous norm $\rho(q_1, \dot{q}_1, q_2)$ w.r.t. dilation $\tilde{r} = (\underbrace{1,1,\cdots,1}_{2n}, \bar{r})$ can be viewed as a homogeneous norm $\hat{\rho}(x_1, x_2) = \rho([z_1^T \; z_2^T]^T, [\dot{z}_1^T \; \dot{z}_2^T]^T, q_2)$ w.r.t. $\Delta_{\varsigma}^{\hat{r}}$ with $\hat{r} = (\underbrace{1,\ldots,1}_{2\bar{n}_1}, r) \in \Re^{n_1+n_2}$ where $n = \bar{n}_1 + \bar{n}_2$, $q_1 = [z_1{}^T z_2{}^T]^T$, $x_1 = [z_1^T \; \dot{z}_1^T]^T$ and $x_2 = [q_2^T \; z_2^T \; \dot{z}_2^T]^T$. Notice that the two matrices given in (21) are equal to $kI + A_1 + B_1 K_1$ and $\overline{A}_2(a) + \overline{B}_2 K_2$, respectively. Thus, it remains to show that the controllers given in (8) can be written into the form (22)-(23). Using the fact $x_1 = [z_1^T \; \dot{z}_1^T]^T$ and $K_1 = [K_{11} \; K_{12}]$, it can be directly computed that

$$u_1 = (b - K_1 a)\lambda + K_1 x_1 = (k^2 I - K_{11} + kK_{12})\eta_0 \lambda + K_{11} z_1 + K_{12} \dot{z}_1 .$$

That is to say that u_1 can be written into the form (22). Similarly, using the fact $x_2 = [q_2^T \; z_2^T \; \dot{z}_2^T]^T$ and $K_2 = [K_{21} \; K_{22} \; K_{23}]$, the following equations hold:

$$\Delta_{\lambda}^{sr} K_2 x_2 = [\Delta_{\lambda}^{s\bar{r}} K_{21} \; K_{22} \; K_{23}] \begin{bmatrix} q_2 \\ z_2 \\ \dot{z}_2 \end{bmatrix} = \Delta_{\lambda}^{s\bar{r}} K_{21} q_2 + K_{22} z_2 + K_{23} z_3 .$$

Let $K_{21} = (k_{ij})$. Then, we have

$$\Delta_\lambda^{s\bar{r}} K_{21} = (\lambda^{s_i - \bar{r}_j} k_{ij}) = \begin{bmatrix} \lambda^{-\bar{d}_1} k_{11} & \lambda^{-\bar{d}_2} k_{12} \cdots \lambda^{-\bar{d}_m} k_{1m} \\ \lambda^{-\bar{d}_1} k_{21} & \lambda^{-\bar{d}_2} k_{22} \cdots \lambda^{-\bar{d}_m} k_{2m} \\ \vdots & \cdots \quad \vdots \\ \lambda^{-\bar{d}_1} k_{\bar{n}_2 1} & \lambda^{-\bar{d}_2} k_{\bar{n}_2 2} \cdots \lambda^{-\bar{d}_m} k_{\bar{n}_2 m} \end{bmatrix}.$$

In view of the equation above, it can be seen that $\Delta_\lambda^{s\bar{r}} K_{21} q_2 = K_{21} \Delta_{1/\lambda}^{\bar{d}} q_2$. Hence, we have $\Delta_\lambda^{sr} K_2 x_2 = K_{21} \Delta_{1/\lambda}^{\bar{d}} q_2 + K_{22} z_2 + K_{23} z_3$. Particularly, u_2 can be written in the form (23) by virtual of (8). The global ρ-exponential stability follows from Proposition 1. This completes the proof of the theorem. ■

B. Second form of Caplygin systems

In this subsection, an alternative representation of Caplygin systems will be given. It will be useful in the study of practical systems. An illustrated example will be given in next section.

In the remainder of this paper, we always assume that (C1) holds. Thus, the constraints function J can be written as $J(z_1, z_2) = [\sum_{j=1}^{\bar{n}_2} J_{1j}(z_1) z_{2j} \quad \bar{J}_2(z_1)]$. For each $1 \le j \le \bar{n}_2$, let $\widetilde{J}_{1j}$ be defined as follows:

$$\widetilde{J}_{1j} = J_{1j} - \partial J_{2j} / \partial z_1, \tag{24}$$

where J_{2j} is *j*-th column vector of $\bar{J}_2$. Consider the following coordinate transformation:

$$\widetilde{q}_2 = q_2 + \bar{J}_2(z_1) z_2. \tag{25}$$

Then, we have

$$\begin{aligned} \dot{\widetilde{q}}_2 &= \dot{q}_2 + \bar{J}_2 \dot{z}_2 + \sum_{j=1}^{\bar{n}_2} (\partial J_{2j} / \partial z_1) z_{2j} \dot{z}_1 \\ &= -\sum_{j=1}^{\bar{n}_2} J_{1j} z_{2j} \dot{z}_1 - \bar{J}_2 \dot{z}_2 + \bar{J}_2 \dot{z}_2 + \sum_{j=1}^{\bar{n}_2} (\partial J_{2j} / \partial z_1) z_{2j} \dot{z}_1 = -\sum_{j=1}^{\bar{n}_2} \widetilde{J}_{1j} z_{2j} \dot{z}_1 = -\widetilde{J} \dot{q}_1, \end{aligned}$$

where

$$\widetilde{J}(z_1, z_2) = [\sum_{j=1}^{\bar{n}_2} \widetilde{J}_{1j}(z_1) z_{2j} \quad 0]. \tag{26}$$

In new coordinate $(q_1, \dot{q}_1, \widetilde{q}_2)$, the transformed system is still a Caplygin system in the form (1)-(2).

For the convenience, it can be called as the second form of Caplygin systems. Moreover, (C1) also holds by (26). Then, theorem 1 can be used to study the exponential stabilizability for new system. We summarize the previous discussions into the following proposition.

Proposition 3. Consider a Caplygin system of the form (1)-(2). Suppose (C1) holds. Using the new coordinate $(q_1, \dot{q}_1, \tilde{q}_2)$ with $\tilde{q}_2$ being defined as (25), the new system called as second form is also a Caplygin system of the form (1)-(2) and (C1) still holds where the new constraint function can be described as (24) and (26). ■

Remark 2. It may be guessed that (C2) is invariant under various coordinate transformations since it is a controllability condition. Unfortunately, it is not true in general. In fact, we will show that the rolling wheel as a Caplygin system does not satisfy (C2) in its original coordinate representation, but using the coordinate transformation described above, (C2) becomes to be true for the new system in next section. That is to say that ***(C2) is a condition depending on coordinate transformations***.

IV. A Simple Degree Criterion and Examples

In this section, Theorem 1 will be used to study an important class of Caplygin systems. Again we assume that (C1) holds. Throughout this section, we assume that $\bar{n}_2 = 1$. Under this assumption, a simple criterion can be proposed to check (C2) as follows.

Proposition 4. Let $\bar{d}_i$ be the constant defined in (12) for each $1 \le i \le m$. Let $\bar{P}(\eta) = (a_1(\eta), a_2(\eta), \cdots, a_m(\eta))^T, \forall \eta \in \Re^{n_1}$. Assume that (C1) holds and $\bar{n}_2 = 1$. Then, (C2) holds if and only if the following conditions hold.

(a) There exists a vector $\eta_0 \in \Re^{n_1}$ so that $a_i(\eta_0) \neq 0, \forall 1 \le i \le m$.

(b) $\bar{d}_i \neq \bar{d}_j, \forall i \neq j$.

In addition to $J_{11} \equiv 0$, (C2) is equivalent to the following conditions.

(c) There exists a vector $\eta_0 \in \Re^{n_1}$ so that $p^{J}_{i(\bar{n}_1+1)}(\eta_0) = p^{\bar{J}_2}_{i1}(\eta_0) \neq 0, \forall 1 \le i \le m$.

(d) $d^J_{i(\overline{n}_1+1)} = d^{\overline{J}_2}_{i1} \neq 0, \forall 1 \le i \le m,\ and\ d^J_{i(\overline{n}_1+1)} \neq d^J_{j(\overline{n}_1+1)}, \forall i \neq j$.

Proof. Let $\overline{r}_i = \overline{d}_i + 1$ as defined in (12), $\forall 1 \le i \le m$. It is straightforward to see that that $D_{\overline{r}}\overline{P}(\eta) = (\overline{r}_1 a_1(\eta), \overline{r}_2 a_2(\eta), \cdots, \overline{r}_m a_m(\eta))^T, \forall \eta \in \Re^{n_1}$. Thus, the determinant of the controllability-like matrix given in (C2) can be explicitly computed as follows:

$$\det\left(\left[\overline{P}, D_{\overline{r}}\overline{P}, \cdots, D_{\overline{r}}^{m-1}\overline{P}\right]\right) = \det\begin{bmatrix} a_1 & \overline{r}_1 a_1 & \cdots & \overline{r}_1^{m-1} a_1 \\ a_2 & \overline{r}_2 a_2 & \cdots & \overline{r}_2^{m-1} a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & \overline{r}_m a_m & \cdots & \overline{r}_m^{m-1} a_m \end{bmatrix} = \left(\prod_{i=1}^m a_i\right)\det\begin{bmatrix} 1 & \overline{r}_1 & \cdots & \overline{r}_1^{m-1} \\ 1 & \overline{r}_2 & \cdots & \overline{r}_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{r}_m & \cdots & \overline{r}_m^{m-1} \end{bmatrix}$$
$$= \left(\prod_{i=1}^m a_i\right)\prod_{1\le i<j\le m}(\overline{r}_i - \overline{r}_j) = \left(\prod_{i=1}^m a_i\right)\prod_{1\le i<j\le m}(\overline{d}_i - \overline{d}_j).$$

by using the property of Vandermode matrix. This implies that (C2) holds if and only if conditions (a) and (b) holds. Since $\overline{n}_2 = 1$ and $J_{11} \equiv 0$, we have $J = [0\ \ \overline{J}_2]$. In this case, we have $\overline{d}_i = d^{\overline{J}_2}_{i1} = d^J_{i(\overline{n}_1+1)}, \forall 1 \le i \le m,$ by the definition. Thus, it can be directly checked that $\overline{P}_2 = (p^J_{1(\overline{n}_1+1)}, p^J_{2(\overline{n}_1+1)}, \cdots, p^J_{m(\overline{n}_1+1)})^T$ in view of (15). Moreover, the matrix-valued function $\overline{P}$ can be computed as $\overline{P} = -D_{\overline{d}}\overline{P}_2 = (-\overline{d}_1 p^J_{1(\overline{n}_1+1)}, -\overline{d}_2 p^J_{2(\overline{n}_1+1)}, \cdots, -\overline{d}_m p^J_{m(\overline{n}_1+1)})^T$. Then, (a) is equivalent to (c) and $\overline{d}_i \neq 0, \forall 1 \le i \le m$. It finishes the proof of the proposition by the previous discussions. ■

In the following, three illustrated examples are given based on Theorem 1 and Proposition 4.

Example 1. **(Knife edge).** Consider a knife edge system as follows [2]:

$$\begin{aligned} \dot{y}_1 &= y_4, \\ \dot{y}_2 &= y_5, \\ \dot{y}_3 &= -y_1 y_5, \\ \dot{y}_4 &= v_1 + v_2 y_3 - y_1 y_5^2, \\ \dot{y}_5 &= v_2, \end{aligned} \tag{27}$$

where y_i is state variable, $\forall 1 \le i \le 5$, and v_j is control variable, $\forall 1 \le j \le 2$. Consider the following coordinate transformation:

$$z_1 = y_1, z_2 = y_5, q_1 = [z_1, z_2]^T, q_2 = y_3, u_1 = v_1 + v_2 y_3 - y_1 y_5^2, u_2 = v_2, u = [u_1, u_2]^T. \tag{28}$$

Then, the system (27) can be transformed into a Caplygin system of the form (1)-(2) with $m=1$ and the constraint function $J = [0\ z_1]^T$. Moreover, (C1) holds with $\bar{n}_1 = 1$, $\bar{n}_2 = 1$, $J_{11} \equiv 0$ and $\bar{J}_2 = z_1$. Thus, it only needs to verify conditions (c)-(d) in Proposition 4 to exponentially stabilize the systems (27). Since $m=1$ and $d_{12}^J = d_{11}^{\bar{J}_2} = 1$, it is easy to see that condition (d) holds. Furthermore, $p_{12}^J(\eta_0) = p_{11}^{\bar{J}_2}(\eta_0) = \eta_0 \neq 0, \forall \eta_0 \neq 0$. Thus, condition (c) also holds and the origin is globally ρ-exponentially stabilizable according to Theorem 1 and Proposition 4. ■

Example 2. **(Extended power form).** Consider an extended power form system as follows [6]:

$$\begin{aligned}
&\ddot{y}_1 = u_1, \\
&\dot{y}_{\bar{n}+1} = \dot{y}_2 y_1^{\bar{n}-1}/(\bar{n}-1)!, \\
&\quad\vdots \\
&\dot{y}_3 = \dot{y}_2 y_1, \\
&\ddot{y}_2 = u_2,
\end{aligned} \tag{29}$$

where y_i and $\dot{y}_j$ are state variable, $\forall 1 \le i \le \bar{n}+1$, $\forall 1 \le j \le 2$, and u_j is control variable, $\forall 1 \le j \le 2$. The system (29) is a Caplygin system of the form (1)-(2) with $q_1 = [y_1, y_2]^T, q_2 = [y_{\bar{n}+1}, \cdots, y_3]^T, u = [u_1, u_2]^T$, $m = \bar{n} - 1$ and the constraint equations $J = [0\quad \bar{J}_2]$ where $\bar{J}_2 = -[y_1^{\bar{n}-1}/(\bar{n}-1)! \cdots y_1]^T$. Notice that condition (C1) holds with $z_1 = y_1$ and $z_2 = y_2$, $\bar{n}_1 = 1$, $\bar{n}_2 = 1$ and $J_{11} \equiv 0$. Thus, it only needs to verify conditions (c)-(d) in Proposition 4 to exponentially stabilize the systems (29). It is easy to see that $d_{i2}^J = d_{i1}^{\bar{J}_2} = \bar{n} - i, \forall 1 \le i \le m = \bar{n} - 1$, in view of the form of $\bar{J}_2$. Hence, condition (d) holds. Moreover, $p_{i2}^J(\eta_0) = p_{i1}^{\bar{J}_2}(\eta_0) = -\eta_0^{\bar{n}-i}/(\bar{n}-i)! \neq 0, \forall 1 \le i \le m = \bar{n}-1, \forall \eta \neq 0.$ Thus, condition (c) also holds and the origin is globally ρ-exponentially stabilizable according to Theorem 1 and Proposition 4. ■

Example 3. **(Rolling wheel).** Consider a rolling wheel system as follows [2]:

$$
\begin{aligned}
\dot{y}_1 &= y_5, \\
\dot{y}_2 &= y_6, \\
\dot{y}_3 &= -y_5 \cos(y_2), \\
\dot{y}_4 &= y_5 \sin(y_2), \\
\dot{y}_5 &= v_1 / 2, \\
\dot{y}_6 &= v_2,
\end{aligned}
\tag{30}
$$

where y_i is state variable, $\forall 1 \le i \le 6$, and v_j is control variable, $\forall 1 \le j \le 2$.

Consider the following coordinate transformation:

$$
z_1 = y_2, z_2 = y_1, q_1 = [z_1, z_2]^T, q_2 = [y_3, y_4]^T, u_1 = v_2, u_2 = v_1 / 2, u = [u_1, u_2]^T. \tag{31}
$$

Then, the system (30) can be transformed into a Caplygin system of the form (1)-(2) with $m = 2$ and the constraint functions

$$
J = \begin{bmatrix} 0 & \cos(z_1) \\ 0 & -\sin(z_1) \end{bmatrix}.
$$

Thus, (C1) holds with $\overline{n}_1 = 1$, $\overline{n}_2 = 1$, $J_{11} \equiv 0$ and $\overline{J}_2 = [\cos(z_1), -\sin(z_1)]^T$. To apply Theorem 1, it is necessary to verify conditions (c)-(d) in Proposition 4. However, the condition (d) in Proposition 4 does not hold since $p_{12}^J = p_{11}^{\overline{J}_2} = 1$ and $d_{12}^J = d_{11}^{\overline{J}_2} = 0$. Alternatively, let us try to very (C2) for the second form of Caplygin system described in subsection III.B. According to (24), the new constrain function $\widetilde{J} = [\widetilde{J}_{11} z_2 \quad 0]$ where

$$
\widetilde{J}_{11} = J_{11} - \partial \overline{J}_2 / \partial z_1 = \begin{bmatrix} \sin(z_1) \\ \cos(z_1) \end{bmatrix}.
$$

In this case, $\overline{P}_2 \equiv 0$ and $\overline{J}_1(z_1, z_3) = \widetilde{J}_{11}(z_1) z_3 = [\sin(z_1) z_3, \cos(z_1) z_3]^T$, $\forall z_1 \in \Re, \forall z_3 \in \Re$. Thus, $(p_{i1}^{\overline{J}_1}) = [z_1 z_3, z_3]^T$ and $\overline{d} = (\overline{d}_1, \overline{d}_2) = (d_{11}^{\overline{J}_1}, d_{21}^{\overline{J}_1}) = (2,1)$. Then,

$$
\overline{P}(\eta) = \overline{P}_1(\eta) - (D_{\overline{d}}) \overline{P}_2(\eta) = \overline{P}_1(\eta) = (\overline{p}_{i1}^1(\eta, \eta)) = (p_{i1}^{\overline{J}_1}(\eta, \eta)) = [\eta^2, \eta]^T, \quad \forall \eta \in \Re.
$$

Particularly, conditions (a)-(b) in Proposition 4 hold for any $\imath_0 \neq 0$. Thus, the origin is globally ρ-exponentially stabilizable for new coordinated system by Theorem 1 and Proposition 4. ■

As a final example, let us consider the set-point problem for a hopping robot system.

Example 4. **(Hopping robot).** Consider a hopping robot system as follows [9]:

$$\begin{aligned} \dot{\psi} &= a, \\ \dot{l} &= v, \\ \dot{\theta} &= -\frac{m_l(l+1)^2}{1+m_l(l+1)^2}\omega, \end{aligned} \tag{32}$$

where (ψ, l, θ) denote the body angle, leg extension, and leg angle of the robot; m_l is the mass of the leg at the foot; a and v are the velocities of ψ and l, respectively; Let τ and T denote the torque and the force w.r.t. ψ and l, respectively. Then, we have

$$\begin{aligned} \dot{a} &= \tau / J, \\ \dot{v} &= F / M, \end{aligned} \tag{33}$$

where J and M represents the inertial mass and the mass, respectively.

The so-called set-point problem is to find a controller (τ, F) so that every trajectory (ψ, l, θ, a, v) of system (32)-(33) converges to a specific target $(\psi, l, \theta, a, v) = (\psi_0, l_0, \theta_0, 0, 0)$ for some non-negative constant l_0. Define the following error variables and control variables as follows:

$$z_1 = l - l_0, z_2 = \psi - \psi_0, q_1 = [z_1, z_2]^T, q_2 = \theta - \theta_0 + (\psi - \psi_0)m_l(l_0+1)^2/[1+m_l(l_0+1)^2], \tag{34}$$

and $u_1 = F/M, u_2 = \tau/J, u = [u_1, u_2]^T$. (35)

Thus, the set-point problem is reduced to a stability problem since $\lim_{t\to\infty}(q_1(t), q_2(t), \dot{q}_1(t)) = 0$ is equivalent to $\lim_{t\leftarrow\infty}(\psi(t), l(t), \theta(t), a(t), v(t)) = (\psi_0, l_0, \theta_0, 0, 0)$. Moreover, the error system can be transformed into a Caplygin system of the form (1)-(2) with $m = 1$ and the constraint functions

$$J = \begin{bmatrix} 0 & \bar{J}_2(z_1) \end{bmatrix},$$

where $\bar{J}_2 = \dfrac{m_l(z_1+l_0+1)^2}{1+m_l(z_1+l_0+1)^2} - \dfrac{m_l(l_0+1)^2}{1+m_l(l_0+1)^2}$ and (C1) holds with $\bar{n}_1 = 1$, $\bar{n}_2 = 1$, $J_{11} \equiv 0$. Notice that $p_{12}^J = p_{11}^{\bar{J}_2} = 2m_l(l_0+1)/[1+m_l(l_0+1)^2]^2 z_1$ and thus $d_{12}^J = d_{11}^{\bar{J}_2} = 1$. It is straightforward to see that conditions (c)-(d) holds. Then, the origin of the error system is globally ρ-exponentially stabilizable according to Theorem 1 and Proposition 4. That is to say that the set-point control problem can be solved via the controller (22)-(23) by employing Theorem 1. ■

Remark 3. Example 4 particularly shows that the set-point problem for Caplygin systems can also be solved by considering the error systems and employing Theorem 1. Due to a limited space, a formal description is omitted. ■

V. Conclusions

The ρ-exponential stabilizability of nonholonomic Caplygin systems was studied. The global ρ-exponential stabilizability of the origin was guaranteed based on a simplified rank condition. The proposed criterion is easily checked and simpler than the result given in [7]. An interesting coordinate transformation (second form) of Caplygin systems was also given so that the proposed criterion can be applied to various situations. For the case of $\overline{n}_2 = 1$, the rank condition was further reduced to some conditions relating to the degree and non-zero property of the lowest polynomials of constraint functions. Several illustrated examples were given to validate the effectiveness of our approaches. The future work may toward to deduce a similar result for a more general class of nonholonomic systems. In this direction, the results proposed in [11], can be served as a guiding line. On the other hand, the robustness for the proposed controllers is also interesting and deserves more discussion in view of the recent result given in [5].

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