中央研究院
資訊科學研究所
Institute of Information Science, Academia Sinica • Taipei, Taiwan, ROC

TR-IIS-17-002

THE PROJECTIVE LINE AS A MERIDIAN

Kelly McKennon

May 10, 2017 || Technical Report No. TR-IIS-17-002

http://www.iis.sinica.edu.tw/page/library/TechReport/tr2017/tr17.html

THE PROJECTIVE LINE AS A MERIDIAN

(by Kelly McKennon, May 2017)

Prologue

This paper is written in $\TeX$ `eplain`, for compilation using `pdf.tex`. This was done so that links and colors would be available without the restrictions of LaTex.

There are some symbols used here which are not universally standard, all of which should be covered in the appendix. The most common are the following:

$$\mathrm{X} \triangle \mathrm{Y}$$

for the set complement of a subset Y of a set X;

$$\llcorner\phi\lrcorner \quad \text{and} \quad \ulcorner\phi\urcorner\text{, respectively,}$$

for the domain and range, respectively, of a function ϕ;

$$\overrightarrow{\phi}(\mathrm{S}) \equiv \{\phi(\mathrm{x}) : \mathrm{x} \in \mathrm{S}\}$$

for the image of a subset S of the domain of a function ϕ;

$$\underline{\mathrm{n}} \equiv \{1,2,\ldots n\}$$

for the set of the first n positive integers and

$$\mathrm{X}^{\mathrm{Y}}$$

for the family of all functions from a set Y to a set X.

Some of the material in these papers appeared originally in http://vixra.org/abs/1306.0233 , and several mistakes in that paper have been rectified here.

The author can be reached at KellyMack@Proton.Com .

Table of Contents

1. Introduction

(1.1) Purpose The purpose of the present article is to examine the essence of what has commonly been described as a "projective line", but which is here named a "meridian". This shall be done in several papers: this first paper devoted to the meridian itself, the second to the character and form of the family of projective isomorphisms of one meridian onto another and the third to some connections between meridians and higher dimensional projective space.

In this first paper we shall view the meridian from various points of view:

(1) as a set acted upon by a family of involutions;

(2) as a set acted upon by a 3-transitive group of permutations;

(3) as a set with a quinary operator;

(4) as an equivalence class of quadruples, relating to the cross ratio.

In the final section of this first paper we shall investigate how the existence of a certain single-valued exponential on a meridian is characteristic of the meridian corresponding to the field of real numbers.

Most of the terminology applied here is standard, but not all. Both standard and non-standard terminology is detailed in the appendix. This appendix is somewhat more encompassing than necessary for what is required here, so that it can serve for the sequel as well. There is also an index for notation as well as the index for terminology.

(1.2) Historical The first considerations of perspective are perhaps coeval with the development of man's eyesight, and records of such are traceable to antiquity.[1]

Some of the Renaissance painters and architects tried to organize what was then known about perspective to aid them in their art: *exempli gratia*, Filipo Brunelleshi (1337-1446), Leone Battista Alberti (1404-1472), Piero della Francesca (1410-1492), Leonardo Da Vinci (1452-1519) and Albrecht Dürer (1471-1528).

A crucial intellectual step in understanding perspective is the conception of points at "infinity". This surfaced in the early seventeenth century, and manifested itself, probably independently, through the minds of Johannes Kepler (1571-1630) and Gérard Desargues (1591-1661). The work of the latter inspired the young Blaise Pascal (1623-1669) to write in 1639 a significant treatise on projective geometry, the mathematical formalization of the ideas of perspective.

It was in the early nineteenth century, along with the rapid and accelerating progress of nearly all science at the time, that interest and understanding in projective geometry again surfaced: *exempli gratia*, through Gaspard Monge (1746-1818) at the turn of the century, Jean-Victor Poncelet (1788-1867) in 1822 and Jakob Steiner (1796-1863) in 1833.

The interest in perspective, by its very nature, is geometrical. Another mathematical interest, of which the published roots are traceable even further back into antiquity than the study of perspective, is the solution of algebraic equations.[2] Perhaps the most brilliant and influential work on the subject was done by Ëvariste Galois (1811-1832) during the last three years of his short life, when he developed and published what afterwards came to be known as "Galois theory".[3] Several concepts which later were to loom large

---

[1] A famous projective theorem concerning hexagons was published by Pappus of Alexandria in the first half of the fourth century AD.

[2] Evidence of such interest is present in Babylonian clay tablets before 2000BC.

[3] Galois died in a dual and, foreseeing his possible death, wrote a famous letter to Auguste Chevalier describing his ideas for solving equations. The celebrated mathematician Hermann Weyl once said, "This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole history of mankind."

(1.5) Projective Isomorphisms If X and Y are two projective spaces, we say that a bijection $\phi \big| X \hookrightarrow Y$ is a **projective isomorphism** if

$$\mathfrak{Homograph}(Y) = \{\phi \circ \eta \circ \phi^{-1} : \eta \in \mathfrak{Homograph}(X)\}. \tag{1}$$

When X = Y, a projective isomorphism is sometimes called a **projective automorphism**. It is easy to see and to show that

$$(\forall\, \eta \in \mathfrak{Homograph}(X)) \quad \eta \text{ is a projective automorphism of } X. \tag{2}$$

Sometimes the converse of (2) holds and sometimes it doesn't.

It is evident that projective isomorphisms are collineations. If a finite dimensional projective space X is not 1-dimensional, it can be shown that collineations are projective isomorphisms.

(1.6) Projective Automorphisms from Homogeneous Coordinates If β is a vector representation of a projective space X, we shall say that the field of the vector space is a **representation field** of X. One can show that any two representation fields of a projective space are isomorphic as fields. There is a description of the general form of a projective automorphism of a projective space X in terms of the automorphisms of the representation fields of X. To present this description, we shall introduce the idea of "homogeneous coordinates" for projective spaces.

Let F be a field (of characteristic different from 2) and $\mathfrak{n}$ a natural number. We shall say that a subset S of $\overbrace{F \times \ldots \times F}^{\text{n times}}$ is $\mathfrak{n}$-**homogeneous** if no element of S has each coordinate zero,

$$(\forall\, [s_1,\ldots,s_\mathfrak{n}] \in S)(\forall\, k \in F) \quad [k \cdot s_1,\ldots,k \cdot s_\mathfrak{n}] \in S \tag{1}$$

and

$$(\forall\, \{[s_1,\ldots,s_\mathfrak{n}],[t_1,\ldots,t_\mathfrak{n}] \subset S\})(\exists\, k \in F) \quad [k \cdot s_1,\ldots,k \cdot s_\mathfrak{n}] = [t_1,\ldots,t_\mathfrak{n}]. \tag{2}$$

We shall write the family of all $\mathfrak{n}$-homogeneous subsets of $\overbrace{F \times \ldots \times F}^{\text{n times}}$ by

$$F^{(\mathfrak{n})}. \tag{3}$$

Now let V be a vector space over the field F of dimension $\mathfrak{n}$. We shall denote the family of all lines through the origin o of V by

$$V_{\mathfrak{proj}}. \tag{4}$$

Let $[b_1,\ldots,b_n]$ is a basis for the vector space. Then there are $\mathfrak{n}$ unique functions $\lambda_i \big| V \hookrightarrow F$ such that, for each $x \in V$,

$$x \;=\; \lambda_1(x) \cdot b_1 + \ldots + \lambda_\mathfrak{n}(x) \cdot b_n. \tag{5}$$

We define the bijection

$$\nu_\mathfrak{n} \big| V_{\mathfrak{proj}} \ni L \hookrightarrow \{[\lambda_1(x),\ldots,\lambda_\mathfrak{n}(x)] : x \in L\} \in F^{(\mathfrak{n})}. \tag{6}$$

Now let α be a field automorphism of F. We define $\alpha^{(\mathfrak{n})} \big| F^{(\mathfrak{n})} \hookrightarrow F^{(\mathfrak{n})}$ by

$$(\forall\, S \in F^{(\mathfrak{n})}) \quad \alpha^{(\mathfrak{n})}(S) \;\equiv\; \{[\alpha(s_1),\ldots,\alpha(s_\mathfrak{n})] : [s_1,\ldots,s_\mathfrak{n}] \in S\} \tag{7}$$

Thus, relative to the given basis, each field automorphism α acts on the family of lines through the origin by

$$(\nu_\mathfrak{n})^{-1} \circ \alpha^{(\mathfrak{n})} \circ \nu_\mathfrak{n} \big| V_{\mathfrak{proj}} \hookrightarrow V_{\mathfrak{proj}}. \tag{8}$$

over the landscape of mathematics, were inherent in Galois' work: the concepts of a "group" and a "field". Niels Henrik Abel (1802-1829), a contemporary of Galois doing important work on the solution of equations, also implicitly used the concept of a field in his work.[4]

The algebraic concept of a field is intimately related to the geometric concept of a projective space. This was brought to light in a 1857 paper by Karl Georg Christian von Staudt (1798-1867). In his landmark book Geometrie der Lage published in 1847, von Staudt had already laid down the first rigorous axiom system for projective geometry, stripping away the superfluous notions of length and angle, and drawing attention to the fundamental notions of harmonic conjugates and polarity which animate the symmetry forming the heart of the subject. In his 1857 paper he introduced the concept of a *Wurf*[5] of which we shall have more to say *infra*. These *Wurf*s[6] lead to the construction of a field and, conversely, every field arises in such a way from some projective space. This foreshadowed a future wherein the study of projective spaces was to proceed along two parallel paths, one employing the manipulative tools of algebra, and the other the visual figures of geometry.

(1.3) Definition of Projective Space The algebraic path has as its foundation (along with the concept of a field) the construct of a "vector space". Although the inherent ideas had been around since the early eighteenth century, the formal definition as it is today was given by Giuseppe Peano (1858-1932) in 1888. Vector spaces permeate much of present day mathematics, as well as physics and engineering. Because of their ubiquity and familiarity, the temptation to adopt them as the vehicle of projective geometry is rather strong.

Here is one of several possible equivalent definitions. We consider a set X and a group

$$\mathfrak{Homograph}(\mathrm{X}) \tag{1}$$

of permutations of X such that there exists a vector space V over a field F and a bijection β of X onto the family of lines through the origin o of V such that

$$\mathfrak{Homograph}(\mathrm{X}) = \{\beta^{-1}\circ\overrightarrow{\phi}\circ\beta \,\big|\, \mathrm{X}\ni \mathrm{x}\hookrightarrow \beta^{-1}(\{\phi(\mathrm{t}):\mathrm{t}\in\beta(\mathrm{x})\})\in \mathrm{X} : \phi \text{ is a linear automorphism of } \mathrm{V}\,.\} \tag{2}$$

Thus the permutations in $\mathfrak{Homograph}(\mathrm{X})$ are induced by the vector space automorphisms of V applied to the lines through the origin of V. We shall say that X is a **projective space with defining family of homographies** $\mathfrak{Homograph}(\mathrm{X})$. The function β of (2) will be said to be a **vector representation** of the projective space X.

(1.4) Lines and Collineations Suppose that S is a subset of a projective space X. It is easy to show that if β and γ are two vector representations of X, then $\overrightarrow{\beta}(\mathrm{S})$ is the set of lines of a 2-dimensional subspace of the vector space if, and only if, $\overrightarrow{\gamma}(\mathrm{S})$ is as well.[7] Such sets S are said to be **lines** of X. A bijection from one projective space X onto another Y is said to be a **collineation** if it sends the lines of X to the lines of Y. If a projective space is itself a line, it is said to be 1-dimensional.

[4] This is another example of an idea whose time had come. However it was to be in use for about 60 years before it was finally formalized by Heinrich Martin Weber (1842-1913) in 1893. The English term "field" was coined in the same year by Eliakim Hastings Moore (1863-1932). The German word *Körper* (meaning body or corpus), probably more apt, was introduced by Richard Dedekind (1831-1916) in 1871, as a common term for the two fields of real and complex numbers.

[5] or "throw" in English.

[6] or "Würfe" in German.

[7] By definition $\overrightarrow{\beta}(\mathrm{S}) = \{\beta(\mathrm{x}):\mathrm{x}\in\mathrm{S}\}$. *Cf.* (9.2.10).

Let β be a vector representation of a projective space X on V. For each $x \in X$, the set $\nu_n \circ \beta(x)$ is said to be a **set of homogeneous coordinates for** x. When α is a field automorphism of the field F of the vector space V, the mapping

$$\beta^{-1} \circ (\nu_n)^{-1} \circ \alpha^{(n)} \circ \nu_n \circ \beta \tag{9}$$

can without difficulty be shown to be a projective automorphism. We shall call it the **projective automorphism associated with field automorphism α and the vector representation β**. In the case in which X is not 1-dimensional, this projective automorphism is frequently called an **automorphic collineation associated with field automorphism α**.

(1.7) Anatomy of a Projective Automorphism It can be proved that every projective automorphism of a projective space X is the composition of a homography and a projective automorphism associated with a field automorphism. This fact is sometimes viewed in the literature as being a part of the **extended fundamental theorem of projective geometry**. The **first form of the fundamental theorem of projective geometry**, or what we in these papers call the **fundamental theorem of projective geometry** is as follows:

(1.8) The Fundamental Theorem Let X be a projective space. For any two distinct points x and y in X, there is exactly one line

$$\overleftrightarrow{x,y} \tag{1}$$

containing those two points. A subset S of X is called a **projective subspace of** X if, for each $\{x,y\} \subset X$, either $x = y$ or the line $\overleftrightarrow{x,y}$ is a a subset of S. The intersection of all subspaces containing a subset $A \subset X$ will be written

$$A^{oo} \tag{2}$$

and is called the **subspace** X **spanned by** A or, more simply, the **span of** A. A subset S of X is said to be **independent** if no subset of S has the same span as any of its proper subsets. A maximal independent subset is called a **simplex**. A **basis** for X is a subset B of X such that the complement in B of every singleton is a simplex. The **dimension** of a projective space is 1 less than the cardinality of a simplex. Thus the dimension of a line is 1, and so 3 is the cardinality of a basis for a line. What follows is the **fundamental theorem**:

Let $\{a_1,\ldots,a_m\}$ and $\{b_1,\ldots,b_n\}$ be bases for X — then

$$m = n \quad \text{and} \quad (\exists!\, \phi \,|\, S \hookrightarrow S \text{ a homography}) \quad b_1 = \phi(a_1),\ b_2 = \phi(a_2), \ldots,\ b_n = \phi(a_n)\,. \tag{3}$$

(1.9) Perspectivities Let X be a projective space projectively isomorphic with a projective subspace[8] of a projective space Y of one dimension greater than X. Let β and γ be projective isomorphisms of X onto distinct subspaces of Y. Let p be any element of Y which lies on neither of the ranges $\ulcorner\beta\urcorner$ nor $\ulcorner\gamma\urcorner$. Given any point $x \in X$, the line $\overleftrightarrow{p,\beta(x)}$ intersects $\ulcorner\gamma\urcorner$ at exactly one point $\overleftrightarrow{p,\beta(x)} \wedge \ulcorner\gamma\urcorner$. The function

$$X \ni x \hookrightarrow \gamma^{-1}\left(\overleftrightarrow{p,\beta(x)} \wedge \ulcorner\gamma\urcorner\right) \in X \tag{1}$$

[8] A projective subspace S of a projective space P is actually a projective space itself, where
$$\mathfrak{Homograph}(S) \equiv \{\phi|_S : \phi \in \mathfrak{Homograph}(P) \text{ and } \vec{\phi}(S) = S\}\,.$$

is called a **perspectivity of** X. A composition of perspecitivies is called a **projectivity**. It can be shown that

$$\mathfrak{Homograph}(\mathrm{X}) \; = \; \{\phi \big| \mathrm{X} \hookrightarrow \mathrm{X} : \phi \text{ is a projectivity}\} \, . \tag{2}$$

This fact is also considered a part of the "extended" fundamental theorem of projective geometry, and is integral in showing the equivalence of the definition of projective space given here to the various synthetic definitions.

(1.10) Meridians A projective space of dimension 1 is often called a "projective line". This is because the most well-known example may be viewed as a line, corresponding to the field of real numbers. While suggestive, such a denomination is not optimal – for several reasons. There are other common examples, such as finite projective spaces, and the projective space corresponding to the field of complex numbers, which are not lines. Another reason is that even the one dimensional projective space corresponding to the field of real numbers can be viewed as a circle instead of a line, and in some ways is more suggestively viewed in that manner. Consequently we here adopt a different term for a one dimensional projective space: a **meridian**.[9]

From the point of view of the definition of projective space we have just given, we may regard a meridian as the set of lines passing through the origin of a two dimensional vector space. The synthetic definition is not quite so obvious, as synthetic methods using lines and planes require spaces of dimension larger than 1. We shall review and introduce several alternative methods *infra*.

[9] The word "meridian" literally means "middle of the day". If one were on the plane of the ecliptic at the middle of the day, the sun would be directly overhead. A circle passing twice through the polar axis of the earth and the line directly overhead, when viewed from below, appears as a line overhead. This line, located on the plane "at infinity", is called a "celestial meridian". It of course is relative to its observer, since it always lies directly above the observer. It is a physical example of what is here defined as a meridian.

2. Erlanger Definition I

(2.1) Erlanger Programm Felix Klein (1840-1925), while working at the University of Erlangen-Nürnberg, proposed that geometries be characterized in terms of groups $\mathcal{G}$ of permutations. His idea was that each permutation of a set X amounted to changing that set so as to be viewed from a new perspective. Thus the properties of X, which did not change after applications of these permutations, were the essential and intrinsic properties of X (relative to $\mathcal{G}$). This procedure became known as the Erlanger Programm and has been quite successful in a number of various contexts, some not *per se* involving geometry. We shall utilize this method for our initial definitions of a meridian.

(2.2) Funtion Libras[10] Let X and Y be two sets. For any three bijections α, β and γ of X onto Y, we define

$$[\![\alpha,\beta,\gamma]\!] \equiv \alpha\circ\beta^{-1}\circ\gamma\,. \tag{1}$$

A family $\mathcal{L}$ of bijective functions from X onto Y will be called a **function libra** if

$$(\forall\,\{\alpha,\beta,\gamma\}\subset\mathcal{L})\quad [\![\alpha,\beta,\gamma]\!]\in\mathcal{L}\,. \tag{2}$$

For any family $\mathcal{F}$ of bijective functions from X onto Y, we shall write $[\![\mathcal{F}]\!]$ for the intersection of all function libras from X to Y which contain $\mathcal{F}$ as a subfamily. We say that $[\![\mathcal{F}]\!]$ is the **function libra generated by** $\mathcal{F}$.

If X = Y and the elements of $\mathcal{F}$ are involutions (self-inverse bijections), then

$$[\![\mathcal{F}]\!] = \{\phi_1\circ\ldots\circ\phi_{2n-1} : n\in\mathbb{N} \text{ and } \{\phi_1,\ldots,\phi_{2n-1}\}\subset\mathcal{F}\}\,. \tag{3}$$

In this case $[\![\mathcal{F}]\!]$ may be a group, but not necessarily.

(2.3) Definitions Let M be a set with at least four elements. Let $\mathcal{M}$ be a family of non-trivial involutions[11] of M such that

$$(\forall\,\alpha,\beta\in\mathcal{M})\quad \alpha\circ\beta\circ\alpha\in\mathcal{M}\,; \tag{1}$$

$$(\forall\,\{a,b,c,d\}\subset M\colon\ \{a,c\}\cap\{b,d\}=\emptyset)(\exists!\,\phi\in\mathcal{M})\quad \phi(a)=c \quad\text{and}\quad \phi(b)=d\,; \tag{2}$$

and
$$(\forall\,a,b\in M)\quad \{\phi\in\mathcal{M}:\phi(a)=b\}\ \text{is a function libra of permutations of M.} \tag{3}$$

We shall call such a family $\mathcal{M}$ a **meridian family of involutions of** M, and M will be said to be a **meridian relative to** $\mathcal{M}$.

We note that, because the elements of $\mathcal{M}$ are self-inverse, (3) has the following consequence

$$(\forall\,t\in M)(\forall\,\{\alpha,\beta,\gamma\}\subset\mathcal{M}\colon\ \alpha(t)=\beta(t)=\gamma(t))\quad \alpha\circ\beta\circ\gamma=\gamma\circ\beta\circ\alpha\,. \tag{4}$$

For a, b, c and d as in (2) we write $\boxed{a,c;b,d}$ for the function $\phi\in\mathcal{M}$ satisfying

$$(\forall\,\alpha,\beta\in\mathcal{M})\quad \boxed{a,c;b,d}(a)=c \quad\text{and}\quad \boxed{a,c;b,d}(b)=d\,. \tag{5}$$

[10] This is as special case of a more general "libra" which will be defined *infra*. The choice of the word "libra" is due to the fact that it to some degree is an avatar of a balance or set of scales. We shall have more to say on this in a later article.

[11] Self-inverse permutations of M not equal to ι_M.

Two meridians M_1 and M_2, respectively, relative to meridian families $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, of involutions, are said to be **isomorphic as meridians** if there is a bijection $\gamma | M_1 \hookrightarrow M_2$ such that

$$\{\gamma^{-1} \circ \phi \circ \gamma : \phi \in \mathcal{M}_2\} \;=\; \mathcal{M}_1. \tag{6}$$

We say that $[\![\mathcal{M}]\!]$ (*cf.* (2.2.3)) is the group of **homographies** of M.

By a **meridian basis** we shall mean any subset of a meridian of cardinality 3. By an **ordered meridian basis** we mean an ordered triple [a,b,c] where {a,b,c} has cardinality 3.

(2.4) Example: Field Meridian We recall that a group is a set G with a binary operation $G \times G \ni [x,y] \hookrightarrow x \cdot y \in G$ for which there exists $e \in G$ such that

$$(\forall \{x,y,z\} \subset G)(\exists!\, m \in G) \quad e \cdot x = x \cdot e = x, \quad x \cdot m = m \cdot x = e \quad \text{and} \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z\,. \tag{1}$$

The element e is called the **identity** of $\cdot$ and the element m the **inverse** of x relative to $\cdot$. A group and its binary operation are called **abelian** if $x \cdot y = y \cdot x$ for all $\{x,y\} \subset G$.

We further recall that a field is a set F with one abelian group binary operation + and another binary operation $\cdot$ for which there exists an element 1 of F distinct from the identity 0 of the group operation + such that $\cdot$ is a abelian group operation when restricted to the cartesian product of the complement of {0} in F with itself, and such that[12]

$$(\forall \{x,y,z\} \subset F) \quad x \cdot (y+z) \;=\; (x \cdot y) + (x \cdot z)\,. \tag{2}$$

With a field F we write -x for the inverse of an element x relative to the binary operation +, x^{-1} for the inverse of x relative to the operation $\cdot$, and $\frac{x}{y}$ for $x \cdot y^{-1}$.

Let F be a field such that $1+1 \neq 0$.[13] We shall say that F is a **meridian field**.

Let ∞ be any object not in F and let M be the union of F with the singleton[14] $\{\infty\}$. For $\{a,b,c,d\} \subset F$ such that $a \cdot d \neq b \cdot c$, the **homography** $\boxed{a,b,c,d} | M \hookrightarrow M$ is defined by

$$\boxed{a,b,c,d}(x) \;\equiv\; \begin{cases} \dfrac{a \cdot x + b}{c \cdot x + d} & \text{if } x \in F \text{ and } c \cdot x + d \neq 0; \\ \dfrac{b}{d} & \text{if } x = \infty \text{ and } d \neq 0; \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

Let

$$\mathcal{M} \;\equiv\; \{\, \boxed{a,b,c,\text{-}a} : \{a,b,c\} \subset F \text{ and } b \cdot c + a \cdot a \neq 0\}\,. \tag{4}$$

It is a pedestrian exercise to show that $\mathcal{M}$ is a meridian family of involutions of M

(2.5) Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M, and let $\{0,1,\infty\}$ be a basis for M. Let F be the complement in M of the singleton $\{\infty\}$. We define

$$(\forall \{x,y\} \subset F) \quad x+y \;\equiv\; \boxed{\infty,\infty;x,y}(0) \tag{1}$$

and

12 It is common to define $x \cdot y$ to be 0 if either x or y is 0. With this definition condition (2.4.2) becomes trivial when any of x, y or z equals 0.

13 Thus, F is a field with characteristic different than 2.

14 A **singleton** is any set containing a single element.

$$(\forall\,\{x,y\}\subset F:\ 0\notin\{x,y\})\quad x\cdot y \equiv \boxed{\infty,0;x,y}(1)\,. \tag{2}$$

Then F is a field relative to the operations $+$ and $\cdot$, and it is not of characteristic 2.

Proof. Let x and y be in F. Since $\boxed{\infty,\infty;x{+}y,0}$ leaves ∞ fixed and sends 0 to x+y, if follows from the uniqueness part of (2.3.2) that

$$\boxed{\infty,\infty;x{+}y,0}=\boxed{\infty,\infty;x,y}\,. \tag{3}$$

Since all the constituents of $\boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;x,0}$ leave ∞ fixed, it follows from (2.3.3) that it is in $\mathcal{M}$. We have

$$\boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;y,0}(y)=\boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}(0)=\boxed{\infty,\infty;x,0}(0)=x=\boxed{\infty,\infty;x,y}(y)$$

and so from the uniqueness part of (2.3.2) follows

$$\boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;y,0}=\boxed{\infty,\infty;x,y}\,. \tag{4}$$

We have

$$\boxed{\infty,\infty;x{+}y,0}\ \overset{\text{by (3)}}{=\!=}\ \boxed{\infty,\infty;x,y}\ \overset{\text{by (4)}}{=\!=}\ \boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;y,0}\,. \tag{5}$$

Similar reasoning shows that, if $0\notin\{x,y\}$, then

$$\boxed{\infty,0;x{\cdot}y,1}\ =\ \boxed{\infty,0;x,y}\ =\ \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;y,1}\,. \tag{6}$$

For $\{x,u\}\subset F$ such that $0\neq u$, we define

$$\text{-}x\equiv\boxed{\infty,\infty;0,0}(x)\quad\text{and}\quad u^{-1}\equiv\boxed{\infty,0;1,1}(u)\,: \tag{7}$$

we have

$$x+0=\boxed{\infty,\infty;x,0}(0)=x,\quad u\cdot 1=\boxed{0,\infty;u,1}(1)=u, \tag{8}$$

$$\begin{aligned}(\text{-}x)+x&=\boxed{\infty,\infty;\text{-}x,x}(0)\ \overset{\text{by (5)}}{=\!=}\ \boxed{\infty,\infty;\text{-}x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;x,0}(0)\\&=\boxed{\infty,\infty;\text{-}x,0}\circ\boxed{\infty,\infty;0,0}(x)\ \overset{\text{by (7)}}{=\!=}\ \boxed{\infty,\infty;\text{-}x,0}(\text{-}x)=0,\end{aligned} \tag{9}$$

$$\begin{aligned}u^{-1}\cdot u&=\boxed{\infty,0;u^{-1},u}(1)\ \overset{\text{by (6)}}{=\!=}\ \boxed{\infty,0;u^{-1},1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;u,1}(1)\\&=\boxed{\infty,0;u^{-1},1}\circ\boxed{\infty,0;1,1}(u)\ \overset{\text{by (7)}}{=\!=}\ \boxed{\infty,0;u^{-1},1}(u^{-1})=1\,,\end{aligned} \tag{10}$$

and furthermore, for $\{y,z,v,w\}\subset F$ such that $0\notin\{v,w\}$

$$x+y=\boxed{\infty,\infty;x,y}(0)=\boxed{\infty,\infty;y,x}(0)=y+x,\quad u\cdot v=\boxed{\infty,0;u,v}(1)=\boxed{\infty,0;v,u}(1)=v\cdot u, \tag{11}$$

$$\begin{aligned}x+(y+z)&=\boxed{\infty,\infty;x,y{+}z}(0)\ \overset{\text{by (5)}}{=\!=}\ \boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;y{+}z,0}(0)\ \overset{\text{by (5)}}{=\!=}\\&\boxed{\infty,\infty;x,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;y,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;z,0}(0)\ \overset{\text{by (5)}}{=\!=}\\&\boxed{\infty,\infty;x{+}y,0}\circ\boxed{\infty,\infty;0,0}\circ\boxed{\infty,\infty;z,0}(0)\ \overset{\text{by (5)}}{=\!=}\ \boxed{\infty,\infty;x{+}y,z}(0)=(x+y)+z,\end{aligned} \tag{12}$$

and

$$\begin{aligned}u\cdot(v\cdot w)&=\boxed{\infty,0;u,v{\cdot}w}(1)\ \overset{\text{by (6)}}{=\!=}\ \boxed{\infty,0;u,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;v{\cdot}w,1}(1)\ \overset{\text{by (6)}}{=\!=}\\&\boxed{\infty,0;u,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;v,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;w,1}(1)\ \overset{\text{by (6)}}{=\!=}\end{aligned} \tag{13}$$

$$\boxed{\infty,0;u{\cdot}v,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;w,1}(1) \overset{\text{by (6)}}{=\!=} \boxed{\infty,0;u{\cdot}v,w}(1) = (u{\cdot}v){\cdot}w$$

It follows from (8) through (13) that + and · are abelian binary group operations. It remains to show the distributive property (2.4.2). Let then x, y and z be elements of F such that x and y+z are distinct from 0. We define

$$\theta \equiv \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,\infty;y,z}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;x,1}\,. \tag{14}$$

It follows from (2.3.1) that θ is in $\mathcal{M}$. Direct calculation shows that

$$\theta(\infty) \;=\; \infty\,. \tag{15}$$

We have

$$\begin{gathered}\theta(0) = \boxed{\infty,0;x,l}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,\infty;y,z}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;x,1}(0) = \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,\infty;y,z}\circ\boxed{\infty,0;1,1}(\infty) = \\ \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,\infty;y,z}(0) = \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}(y{+}z) = \\ \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;1,y{+}z}\circ\boxed{\infty,0;1,y{+}z}(y{+}z) = \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;1,y{+}z}(1) \overset{\text{by (6)}}{=\!=} \\ \boxed{\infty,0;x,y{+}z}(1) = x{\cdot}(y{+}z)\,.\end{gathered} \tag{16}$$

Furthermore, since (6) implies that $\boxed{\infty,0;x,y} = \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;y,1}$, we have

$$\begin{gathered}\theta(x{\cdot}y) = \\ \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,\infty;y,z}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;x,1}\circ\boxed{\infty,0;x,y}\circ\boxed{\infty,0;x,y}\circ\boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;y,1}(1) \\ = \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,\infty;y,z}\circ\boxed{\infty,0;y,1}(1) = \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}(z) = \\ \boxed{\infty,0;x,1}\circ\boxed{\infty,0;1,1}\circ\boxed{\infty,0;z,1}(1) \overset{\text{by (6)}}{=\!=} \boxed{\infty,0;x,z}(1) = x{\cdot}z\,.\end{gathered} \tag{17}$$

It follows from (15) and (17) that

$$\theta \;=\; \boxed{\infty,\infty;x{\cdot}y,x{\cdot}z}$$

which implies

$$\theta(0) = \boxed{\infty,\infty;x{\cdot}y,x{\cdot}z}(0) = x{\cdot}y{+}x{\cdot}z\,. \tag{18}$$

From (16) and (18) follows that $x{\cdot}(y{+}z) = x{\cdot}y{+}x{\cdot}z$.

If F were of characteristic 2 and x any element of F, then

$$\boxed{\infty,\infty;x,x}(0) = x{+}x = 0 = \boxed{\infty,\infty;0,0}(0)$$

and so $\boxed{\infty,\infty;x,x}$ and $\boxed{\infty,\infty;0,0}$ agree would agree on the two distinct points ∞ and 0, which by (2.3.2) would imply that $\boxed{\infty,\infty;x,x} = \boxed{\infty,\infty;0,0}$. But then $\boxed{\infty,\infty;0,0}$ would leave each element of M fixed, which would be absurd, since ι_M is not an element of $\mathcal{M}$. Q.E.D.

(2.6) Example: Circle Meridian In (2.4) we extended a field by one point to construct a meridian. One can do a similar thing with a plane to construct a "projective plane".

Let **P** be a euclidean plane. For each line L in **P** we shall add a point $\infty(L)$ to L not in **P**. We do this in such a way that points $\infty(L_1)$ and $\infty(L_2)$ are equal if, and only if, the lines L_1 and L_2 are parallel. The set $\infty(\mathbf{P})$ of all these "points at infinity" is called the **line at infinity** and its union with **P** will be denoted by

$$\mathbb{P}\,. \tag{1}$$

Let C be a circle in **P**. Let x and y be elements of C. By $\overleftrightarrow{x,y}$ we shall mean the line through x and y. If p is any point in $\mathbb{P}\triangle C$,[15] and x is any point in C, then the line $\overleftrightarrow{p,x}$, unless it is tangent to C, intersects C at exactly one other point: we write

$$p_C(x) \tag{2}$$

for this other point. In the case of tangency, we define the value of (2) to be just x. It is evident that each such function p_C thus defined is an involution on C. In fact, it can be shown that the family

$$\mathcal{M}(C) \equiv \{p_C : p\in\mathbb{P}\triangle C\} \tag{3}$$

is a meridian family of involutions. Relative to this family, C is said to be a **circle meridian**.

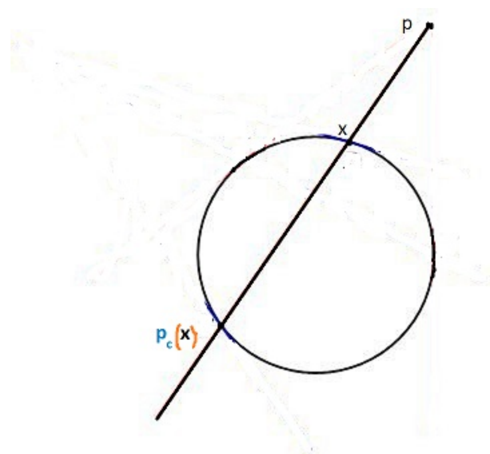


Fig. 1: Circle Meridian Involution

Each circle meridian is isomorphic to every other circle meridian as a meridian.[16]

(2.7) Example: Line Meridian Let $S\equiv\{a,b,c,d\}$ be a subset of $\mathbb{P}$ of cardinality 4. Then the set $\{\overleftrightarrow{a,b},\overleftrightarrow{a,c},\overleftrightarrow{a,d},\overleftrightarrow{b,c},\overleftrightarrow{b,d},\overleftrightarrow{c,d}\}$ is by definition a **complete quadrilateral**. If two lines of this complete quadrilateral intersect S in a common point, they will be said to be **adjacent**: otherwise **opposite**. For instance, the lines $\overleftrightarrow{a,b}$ and $\overleftrightarrow{a,c}$ are adjacent and the lines $\overleftrightarrow{a,b}$ and $\overleftrightarrow{c,d}$ are opposite.

For two distinct lines L and M in $\mathbb{P}$ we shall denote their point of intersection by $L\wedge M$: thus

$$L\cap M = \{L\wedge M\}\,. \tag{1}$$

[15] By $\mathbb{P}\triangle C$ we mean $\{x\in\mathbb{P} : x\notin C\}$. *Cf.* (8.2.15).

[16] The same constructions which have here been adopted for a circle can be adduced for an ellipse, or even for a hyperbola or parabola in $\mathbb{P}$. Each meridian obtained in this manner is isomorphic to a circle meridian.

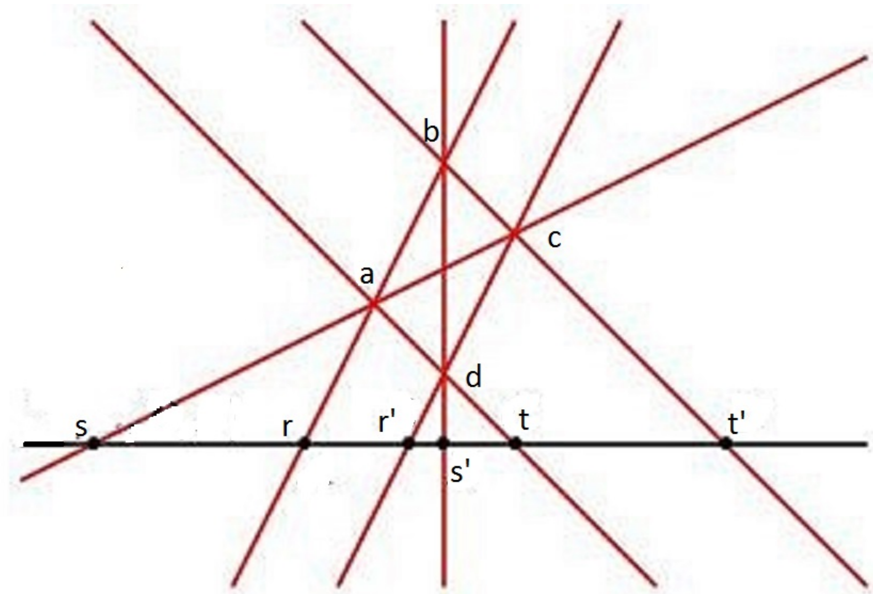


Fig. 2: Cubic Triple of Pairs on a Line

Let L be a line in $\mathbb{P}$ and S as above. If we set $\mathrm{r} \equiv \overleftrightarrow{\mathrm{a,b}} \wedge \mathrm{L}$, $\mathrm{r}' \equiv \overleftrightarrow{\mathrm{c,d}} \wedge \mathrm{L}$, $\mathrm{s} \equiv \overleftrightarrow{\mathrm{a,c}} \wedge \mathrm{L}$, $\mathrm{s}' \equiv \overleftrightarrow{\mathrm{b,d}} \wedge \mathrm{L}$, $\mathrm{t} \equiv \overleftrightarrow{\mathrm{a,d}} \wedge \mathrm{L}$ and $\mathrm{t}' \equiv \overleftrightarrow{\mathrm{b,c}} \wedge \mathrm{L}$, then the set

$$\{\{\mathrm{r,r'}\},\{\mathrm{s,s'}\},\{\mathrm{t,t'}\}\} \tag{2}$$

of pairs of points on L will be called a **cubic triple of pairs of points on L**. It can be shown that cubic triples which agree on five of the six involved points, must agree on all six points. Thus, if {a,b} and {c,d} are disjoint subsets of a line L, then

$$(\forall\, \mathrm{t} \in \mathrm{C})(\exists!\ \boxed{\mathrm{a,b;c,d}}\,(\mathrm{t}) \in \mathrm{L}) \quad \{\{\mathrm{a,b}\},\{\mathrm{c,d}\},\{\mathrm{t},\ \boxed{\mathrm{a,b;c,d}}\,(\mathrm{t})\}\} \text{ is a cubic triple.} \tag{3}$$

It can be shown that the collection of functions $\boxed{\mathrm{a,b;c,d}}$ thus defined form a meridian family $\mathcal{M}(\mathrm{L})$ of involutions of the line L. Relative to this family $\mathcal{M}(\mathrm{L})$, the line L is called a **line meridian**. Every line meridian in $\mathbb{P}$ is isomorphic to every other line meridian.

Furthermore, each line meridian is isomorphic to each circle meridian. If the circle C is tangent to the line L, we can obtain an isomorphism as follows: if p is the point of tangency, choose q in C distinct from p and take the function

$$\mathrm{C} \ni \mathrm{x} \hookrightarrow (\overleftrightarrow{\mathrm{q,x}} \wedge \mathrm{L}) \in \mathrm{L} \tag{4}$$

where $\overleftrightarrow{\mathrm{q,x}} \wedge \mathrm{L} = \infty(\mathrm{L})$ when $\mathrm{x} = \mathrm{q}$.

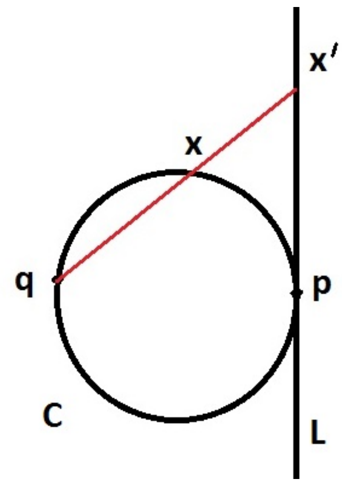


Fig. 3: Meridian Isomorphism from a Circle to a Line

(2.8) Example: Sphere Meridian Let **E** denote three dimensional Euclidian space. As in **P** we associate to each line L a point $\infty(L)$ distinct from **E** in such a way that, for two lines L_1 and L_2, the points $\infty(L_1)$ and $\infty(L_2)$ are equal if, and only if, the two lines are parallel. The set

$$\infty(\mathbf{E}) \equiv \{\infty(L) : L \text{ is a line in } \mathbf{E}\} \tag{1}$$

is called the **plane at infinity**. We denote

$$\mathbb{E} \equiv \mathbf{E} \cup \infty(\mathbf{E}) \tag{2}$$

and call $\mathbb{E}$ a **three dimensional real projective space**. A subset X of $\infty(\mathbf{E})$ is a **line at infinity** if

$$(\exists\, P \text{ a plane in } \mathbf{E}) \quad X = \{\infty(L) : L \text{ is a line in } P\}\,. \tag{3}$$

Let now S be a sphere in **E**. Let L be any line in $\mathbb{E}$ not tangent to S, which intersects S. Then there are two intersection points p and q. If P is the tangent plane at p and Q is the tangent plane at q, then P∩Q is called the **line dual to L relative to the sphere** S. Given any two lines L and M in $\mathbb{E}$ which do not intersect, and any point $x \notin (L \cup M)$, there exists exactly one line

$$\overleftrightarrow{(L,M;x)} \tag{4}$$

which passes through x and intersects both L and M. If L is as above, and x is any point on S, the line $\overleftrightarrow{(L,P\cap Q;x)}$ intersects S at another point

$$\boxed{S;L}\,(x) \tag{5}$$

(where $\boxed{S;L}\,(x)$ is defined to be x if x is on L). The family

$$\mathcal{S} \equiv \{\,\boxed{S;L} : L \text{ a line in } \mathbb{P} \text{ with } \#(L\cap S) = 2\} \tag{6}$$

is a meridian family of involutions of S.[17] The meridian S, relative to this family, will be called a **sphere meridian**.[18]

The field $\mathbb{C}$ of complex numbers, as is well-known, may be associated with the plane **P**, which is then

[17] The expression #(L∩S) means the cardinality of the set L∩S . *Cf.* (9.5.2).

[18] When an ordered basis $[0,\infty,1]$ for S is chosen with 0 and ∞ at opposite ends of a diameter of S and 1 equidistant from 0 and ∞ , S is sometimes called a **Riemann Sphere**.

called the **gaussian plane** $\mathbf{G}$. If we add one point $\infty(\mathbf{G})$, to $\mathbf{G}$, we shall denote it by $\mathbb{G}$. This set $\mathbb{G}$ may be viewed as a meridian in consequence of Theorem (2.5). This meridian is isomorphic to the sphere meridian. There is a classical isomorphism between these two meridians called the **stereographic projection** which is described by the following figure (where p is the point on top of the sphere):

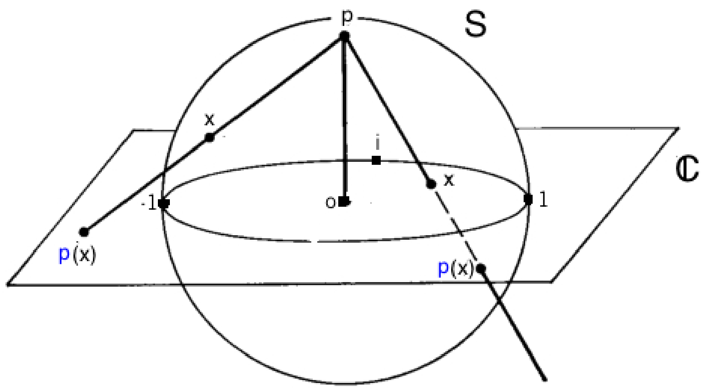


Fig. 4: Stereographic Projection of a Sphere onto the Complex Plane

Here the complex plane runs through the "equator" of the sphere. There is another projection, analogous to the projection of the circle given in (2.7), where the plane is tangent to the sphere.

(2.9) Fundamental Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M. Let $\{a,b,c\}$ and $\{u,v,w\}$ be subsets of M, each of cardinality 3. Then

$$(\exists!\, \phi \in \llbracket \mathcal{M} \rrbracket) \quad \phi(a) = u, \quad \phi(b) = v \quad \text{and} \quad \phi(c) = w\,. \tag{1}$$

Proof. We shall break the existence part of the problem down into the various possible special cases, and verify that (1) holds for each case.

[Case 1: $\{a,b,c\} \cap \{u,v,w\} = \emptyset$] If $d \equiv \boxed{a,u;b,v}\,(c)$ equals w, we let $\phi \equiv \boxed{a,u;b,v}$. Otherwise we let

$$\phi \equiv \boxed{u,v;w,w} \circ \boxed{u,v;d,w} \circ \boxed{a,u;b,v}\,. \tag{2}$$

[Case 2: $a = u$ and $\{b,c\} \cap \{v,w\} = \emptyset$] Same proof as in Case 1.

[Case 3: $a = u$ and $b = v$] Same proof as in Case 1.

[Case 4: $a = v$ and $b = u$] Same proof as in Case 1.

[Case 5: $a = v$ and $\{b,c\} \cap \{u,w\} = \emptyset$] Let $d \equiv \boxed{a,a;c,u}\,(b)$ and then $\phi \equiv \boxed{u,u;a,w} \circ \boxed{u,a;d,w} \circ \boxed{a,a;c,u}$.

[Case 6: $a = v$, $b = w$] Let $\phi \equiv \boxed{a,a;b,u} \circ \boxed{a,b;c,u}$.

[Case 7: $a = v$ and $c = w$] Same proof as in Case 5. All other cases can be subsumed by one of these seven by permuting $\{a,b,c\}$ (and $\{u,v,w\}$ accordingly). Thus existence is shown.

Suppose that there are two functions ϕ and θ sending a to u, b to v and c to w. Then $\phi \circ \theta^{-1}$ fixes all three points a, b and c. Let us rename the ordered basis $[a,b,c]$ as $[o,1,\infty]$ and let F, + and $\cdot$ be as in (2.5). Then $\llbracket \mathcal{M} \rrbracket$ is a family of homographies. Thus there exists $\{p,q,r,s\} \subset F$ such that $\boxed{p,q,r,s} = \phi \circ \theta^{-1}$. We have

$$o = \phi \circ \theta^{-1}(o) = \frac{p \cdot o + q}{r \cdot o + s} \implies q = o\,,$$

$$\infty = \phi \circ \theta^{-1}(\infty) = \frac{p}{r} \implies r = o$$

and

$$1 = \phi \circ \theta^{-1}(1) = \frac{p \cdot 1 + q}{r \cdot 1 + s} = \frac{p}{s} \implies p = s\,.$$

Consequently $\phi\circ\theta^{-1}$ is the identity function and so $\phi=\theta$. Q.E.D.

(2.10) Notation We shall find it useful in the sequel to denote the function ϕ of (2.9.1) by

$$\boxed{\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{u} & \mathrm{v} & \mathrm{w} \end{smallmatrix}}\,. \tag{1}$$

(2.11) Corollary Let $\mathcal{M}$ be a meridian family of involutions of a set M. Let a and b be distinct points of M. Then

$$(\forall\,\phi\in[\![\mathcal{M}]\!]:\ \phi(\mathrm{a})=\mathrm{b}\text{ and }\phi(\mathrm{b})=\mathrm{a})\quad \phi\in\mathcal{M}\,. \tag{1}$$

Proof. Let $\mathrm{c}\in\mathrm{M}$ be distinct from a and b. By the uniqueness part of (2.9), the functions ϕ and $\boxed{\mathrm{a,b;c},\phi(\mathrm{c})}$ are identical. Q.E.D.

(2.12) Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M and let ϕ be a non-involution element of $[\![\mathcal{M}]\!]$. Then there exists $\{\alpha,\beta\}\subset\mathcal{M}$ such that

$$\phi=\alpha\circ\beta\,. \tag{1}$$

Proof. By hypothesis there exists $\mathrm{m}\in\mathrm{M}$ such that m, $\phi(\mathrm{m})$ and $\phi\circ\phi(\mathrm{m})$ are distinct. We have

$$\phi\circ\boxed{\phi(\mathrm{m}),\phi(\mathrm{m});\mathrm{m},\phi\circ\phi(\mathrm{m})}(\phi\circ\phi(\mathrm{m}))=\phi(\mathrm{m})$$

and

$$\phi\circ\boxed{\phi(\mathrm{m}),\phi(\mathrm{m});\mathrm{m},\phi\circ\phi(\mathrm{m})}(\phi(\mathrm{m}))=\phi\circ\phi(\mathrm{m})\,.$$

It follows from (2.11) that $\phi\circ\boxed{\phi(\mathrm{m}),\phi(\mathrm{m});\mathrm{m},\phi\circ\phi}$ is in is in $\mathcal{M}$. Letting $\beta\equiv\boxed{\phi(\mathrm{m}),\phi(\mathrm{m});\mathrm{m},\phi\circ\phi(\mathrm{m})}$ and $\alpha\equiv\phi\circ\boxed{\phi(\mathrm{m}),\phi(\mathrm{m});\mathrm{m},\phi\circ\phi(\mathrm{m})}$, we obtain (1). Q.E.D.

(2.13) Theorem[19] Let M be a meridian with meridian family $\mathcal{M}$ of involutions. Let $[0,1,\infty]$ be an ordered basis for M. Let F be the field associated with this basis as in (2.5). Let α be any field automorphism of F and extend α to all of M by defining $\alpha(\infty)\equiv\infty$. Then

$$\alpha\text{ is a meridian automorphism of M}\,. \tag{1}$$

Furthermore, if ϕ is any meridian automorphism, there exists a basis and field automorphism α as in (1) and an element $\eta\in[\![\mathcal{M}]\!]$ such that

$$\phi=\alpha\circ\eta\,. \tag{2}$$

Proof. Let η be any element of $[\![\mathcal{M}]\!]$ and choose $\{\mathrm{a,b,c,d}\}\subset\mathrm{F}$ such that $\phi=\boxed{\mathrm{a,b,c,d}}$ as in (2.4.3). If $\mathrm{d}=0$, then

$$\alpha\circ\phi\circ\alpha^{-1}(\infty)=\alpha\circ\phi(\infty)=\alpha(\infty)=\infty=\boxed{\alpha(\mathrm{a}),\alpha(\mathrm{b}),\alpha(\mathrm{c}),0}(\infty)=\boxed{\alpha(\mathrm{a}),\alpha(\mathrm{b}),\alpha(\mathrm{c}),\alpha(\mathrm{d})}(\infty)\,. \tag{3}$$

If $\mathrm{d}\neq 0$, then

$$\alpha\circ\phi\circ\alpha^{-1}(\infty)=\alpha\circ\phi(\infty)=\alpha(\frac{\mathrm{b}}{\mathrm{d}})=\frac{\alpha(\mathrm{b})}{\alpha(\mathrm{d})}=\boxed{\alpha(\mathrm{a}),\alpha(\mathrm{b}),\alpha(\mathrm{c}),\alpha(\mathrm{d})}(\infty)\,. \tag{4}$$

If $\mathrm{c}\cdot\mathrm{x}+\mathrm{d}\neq 0$, then

[19] This is the meridian form of the result discussed in (1.7).

$$\alpha\circ\phi\circ\alpha^{-1}(x)=\alpha\circ\phi(\alpha^{-1}(x))=\alpha\left(\frac{a\cdot\alpha^{-1}(x)+b}{c\cdot\alpha^{-1}(x)+d}\right)=\frac{\alpha(a)\cdot x+\alpha(b)}{\alpha(c)\cdot x+\alpha(d)}=\boxed{\alpha(a),\alpha(b),\alpha(c),\alpha(d)}(x)\,. \tag{5}$$

Assertion (1) now follows from (3), (4) and (5).

Let ϕ be any meridian automorphism. Let $q\equiv\phi(0)$, $r\equiv\phi(1)$ and $s\equiv\phi(\infty)$. Let $\alpha\equiv\phi\circ\boxed{\begin{smallmatrix}0&1&\infty\\q&r&s\end{smallmatrix}}^{-1}$. Then α is a meridian automorphism leaving 0, 1 and ∞ fixed. Noting that, for $\{x,y\}\subset F$,

$$\alpha\circ\boxed{\infty,\infty;x,y}\circ\alpha^{-1}(\infty)=\infty \quad\text{and}\quad \alpha\circ\boxed{\infty,\infty;x,y}\circ\alpha^{-1}(\alpha(x))=\alpha\circ\boxed{\infty,\infty;x,y}(x)=\alpha(y)\,,$$

we see that

$$\alpha\circ\boxed{\infty,\infty;x,y}\circ\psi y^{-1}=\boxed{\infty,\infty;\alpha(x),\psi y(y)}$$

whence follows that

$$\alpha(x+y)=\alpha\circ\boxed{\infty,\infty;x,y}(0)=\boxed{\infty,\infty;\psi yyy(x),\psi y(y)}\circ\alpha(0)=\boxed{\infty,\infty;\alpha(x),\alpha(y)}(0)=\psi yy(x)+\alpha(y) \tag{6}$$

Similarly we have

$$\alpha\circ\boxed{0,\infty;x,y}\circ\alpha^{-1}(\infty)=0 \quad\text{and}\quad \psi y\circ\boxed{0,\infty;x,y}\circ\alpha^{-1}(\alpha(x))=\alpha\circ\boxed{0,\infty;x,y}(x)=\alpha(y)\,,$$

and so

$$\alpha\circ\boxed{0,\infty;x,y}\circ\alpha^{-1}=\boxed{0,\infty;\alpha(x),\alpha(y)}$$

whence follows that

$$\alpha(x\cdot y)=\alpha\circ\boxed{0,\infty;x,y}(1)=\boxed{\infty,0;\alpha(x),\alpha(y)}\circ\alpha(1)=\boxed{0,\infty;\alpha(x),\alpha(y)}(1)=\alpha(x)\cdot\alpha(y) \tag{7}$$

From (6) and (7) follows that $\alpha\big|_F$ is a field automorphism. Letting $\eta\equiv\boxed{\begin{smallmatrix}0&1&\infty\\q&r&s\end{smallmatrix}}$, we obtain (2). Q.E.D.

(2.14) Projective Automorphisms of the Sphere Meridian In contrast to the case of the circle meridian, the sphere meridian has projective automorphisms which are not homographies. This is connected with the fact that there is a hausdorff topology inherent in the meridian structure of the circle meridian[20], while there is none inherent in the meridian structure of the sphere meridian. There is a topology on S inherited from the metric on euclidian space E, but this topology cannot be obtained just by examining $\mathfrak{Homograph}(S)$. However if we do apply the particular metric inherited from E to S, then we can describe exactly the meridian automorphisms continuous relative to that topology.[21]

We recall from (2.8) that S induces a duality of lines in $\mathbb{E}$. There is a analogous duality of points with planes as well. Let H be any plane in $\mathbb{E}$ which intersects S in more than a single point. The points of intersection comprise a circle. At each point of that circle there is a plane in $\mathbb{E}$ tangent to S at that point. The intersection $\{h\}$ of all these tangent planes is a singleton, the point of which h is called the **point dual to H relative to S**. This **dual pair** $[H,h]$ induces an involution $\boxed{H;h}$ of which the fixed points are the points on $H\cap S$ and of which the values on other points of S are defined as follows

$$(\forall\, x\in S\triangle(H\cap S))\quad \{\boxed{H;h}(x)\}=\overleftrightarrow{x,h}\cap(S\triangle\{x\})\,. \tag{1}$$

Thus $\boxed{H;h}(x)$ is the point other than x on S through which the line through x and h pierces. The function $\boxed{H;h}$ is evidently continuous and it is not difficult to show that it is a projective automorphism of S. In fact it can be shown that each projective automorphism of S which is continuous relative to the metric

[20] *Cf.* (7.27).

[21] There are however an uncountable number of non-continuous projective automorphisms of S.

inherited from E is of the form $\phi\circ$ [H;h], where ϕ is a homography and H is a plane which intersects S at more than one point.[22]

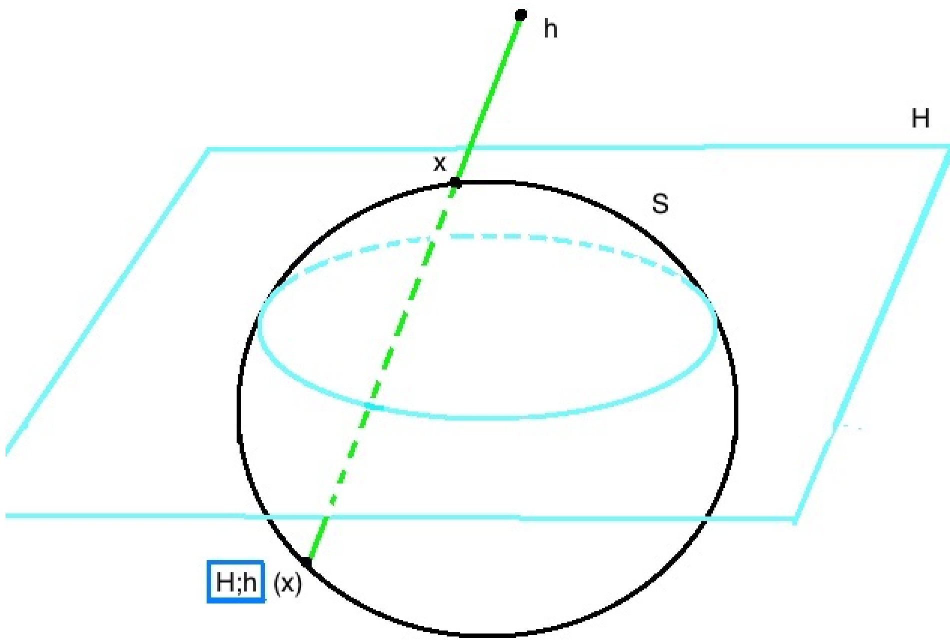


Fig. 5: Continuous Involutive Projective Automorphism

(2.15) Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M. Let ϕ be in $\mathcal{M}$ and suppose that $\phi(\mathrm{m}) = \mathrm{m}$ for some $\mathrm{m} \in \mathrm{M}$. Then there is exactly one other element n of M distinct from m such that $\phi(\mathrm{n}) = \mathrm{n}$.

Proof. There can be no more than one such element n since otherwise it would follow from the fundamental theorem that ϕ would be the identity function on M. Let a be any element of M distinct from m and let $\mathrm{b} \equiv \phi(\mathrm{a})$. If b = a, we may let n ≡ a. Else let n ≡ [a,a;b,b] (m). We have

$$[\mathrm{a,a;b,b}]\circ\phi(\mathrm{a}) = \mathrm{b} \quad \text{and} \quad [\mathrm{a,a;b,b}]\circ\phi(\mathrm{b}) = \mathrm{a}$$

whence from (2.11) follows that [a,a;b,b] $\circ\phi$ is an involution. Consequently

$$[\mathrm{a,a;b,b}]\circ\phi = \phi\circ[\mathrm{a,a;b,b}]. \tag{1}$$

Letting n ≡ [a,a;b,b] (m), we have

$$\phi(\mathrm{n}) = \phi\circ[\mathrm{a,a;b,b}](\mathrm{m}) \overset{\text{by (1)}}{=\!=\!=} [\mathrm{a,a;b,b}]\circ\phi(\mathrm{m}) = [\mathrm{a,a;b,b}](\mathrm{m}) = \mathrm{n}.$$

[22] To this purpose we note that if the automorphism ϕ in (2.13.2) is continuous, then so is the automorphism α which comes from a field automorphism. We choose an ordered basis $[0,1,\infty]$ on S so that the field determined by that basis corresponds to the field isomorphism. It is known that continuous field automorphisms of the complex field are the identity function and complex conjugation. If H is the plane through S containing the three basis elements 0, 1 and ∞, it is not difficult to see that the automorphism [H;h] is just complex conjugation.

Q.E.D.

(2.16) Definitions Let $\mathcal{M}$ be a meridian family of involutions of a set M. A **meridian orbit** is a family $\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\}$ of pairs for which there exists an element $\pi \in \llbracket \mathcal{M} \rrbracket$ such that

$$\pi(a) = b, \quad \pi(b) = c, \quad \pi(c) = d \quad \text{and} \quad \pi(d) = a. \tag{1}$$

Such a function π will be called a **meridian cycle**.

For any $n \in \mathbb{N}$, we define

$$\llbracket M \rrbracket_n \equiv \{\phi \in \llbracket \mathcal{M} \rrbracket : \phi \text{ has exactly } n \text{ fixed points.}\} \tag{2}$$

(2.17) Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M. Then

$$\begin{gathered}(\forall\, \{\{a,b\},\{b,c\}\}:\ \{a,b,c\} \subset M \text{ and } \#\{a,b,c\} = 3)(\exists!\, d \in M) \\ \{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\} \text{ is a meridian orbit.}\end{gathered} \tag{1}$$

Proof. Let $d \equiv \boxed{a,a;c,c}(b)$. Letting $\pi \equiv \boxed{a,b;c,d} \circ \boxed{a,a;c,c}$, direct computation shows that (2.16.1) holds. This proves existence.

For any other π satisfying (2.16.1), $\boxed{a,c;b,b} \circ \boxed{a,b;c,d}$ agrees with π at a, b and d – and so must agree everywhere by the fundamental theorem. It follows that

$$d = \pi(c) = \boxed{a,c;b,b} \circ \boxed{a,b;c,d}(c) = \boxed{a,c;b,b}(d)\,. \tag{2}$$

It follows from (2.15) that b and d are the only fixed points of $\boxed{a,c;b,b}$. Thus d is the unique element of M for which (2.16.1) holds. Q.E.D.

(2.18) Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M. Then

$$(\forall\, \{\{a,b\},\{b,d\}\}:\ \{a,b,d\} \subset M \text{ and } \#\{a,b,d\} = 3)(\exists!\, \tau \in \llbracket \mathcal{M} \rrbracket_1) \quad \tau(d) = d \quad \text{and} \quad \tau(a) = b\,. \tag{1}$$

In addition,

$$\{\{a,b\},\{b,\tau(b)\},\{\tau(b),d\},\{d,a\}\} \text{ is a meridian orbit.} \tag{2}$$

Proof. Since $\boxed{d,d;b,b}$ and $\boxed{d,d;a,b}$ have only the one fixed point d in common, it follows from the fundamental theorem that if they agreed on any other point, they would be equal. It follows that $\boxed{d,d;b,b} \circ \boxed{d,d;a,b}$ is in $\llbracket \mathcal{M} \rrbracket_1$, fixes d and sends a to b. This proves the existence of (1).

Suppose that τ satisfies (1). Let $c \equiv \tau(b)$. Since $\boxed{a,c;b,b} \circ \tau$ interchanges a and b, it follows from (2.11) that it is an involution. Thus

$$d = \boxed{a,c;b,b} \circ \tau \circ \boxed{a,c;b,b} \circ \tau(d) = \boxed{a,c;b,b} \circ \tau \circ \boxed{a,c;b,b}(d) \implies$$

$$\tau(\boxed{a,c;b,b}(d)) = \boxed{a,c;b,b}(d) \implies \boxed{a,c;b,b}(d) =$$

since d is the sole fixed point of τ. Letting $\pi \equiv \boxed{a,c;b,b} \circ \boxed{a,b;c,d}$, direct calculation shows that the set $\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\}$ is a meridian orbit with meridian cycle π. By (2.17.1), it follows that c is unique with this property. Since $\boxed{d,d;b,b} \circ \boxed{d,d;a,b}$ can also serve for τ in the foregoing, we have that

$$\tau(b) = c = \boxed{d,d;b,b} \circ \boxed{d,d;a,b}(b)\,.$$

It follows from the fundamental theorem that $\tau = \boxed{d,d;b,b} \circ \boxed{d,d;a,b}$. Q.E.D.

(2.19) Meridian Harmony Let $\mathcal{M}$ be a meridian family of involutions of a set M and suppose that $\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\}$ is a meridian orbit. Then the pair of pairs $\{\{a,c\},\{b,d\}\}$ is said to be a **harmonic pair**. The following are equivalent assertions about $\{a,b,c,d\} \subset M$:

$$\{\{a,c\},\{b,d\}\} \text{ is a harmonic pair;} \tag{1}$$

$$(\exists\, \alpha \in \mathcal{M}_2) \quad \alpha(a) = c, \quad \alpha(b) = b \quad \text{and} \quad \alpha(d) = d\,; \tag{2}$$

$$(\exists\, \tau \in \llbracket \mathcal{M} \rrbracket_1) \quad \tau(a) = b, \quad \tau(b) = c \quad \text{and} \quad \tau(d) = d\,. \tag{3}$$

Proof. [(1)$\Longrightarrow$(2)] Suppose that (1) holds and let π be as in (2.3.1). Let $\psi \equiv \boxed{a,c;b,b} \circ \pi \circ \boxed{a,c;b,b}$ and $r \equiv \boxed{a,c;b,b}(d)$. We have

$$\psi(c) = r, \quad \psi(b) = c, \quad \psi(a) = b \quad \text{and} \quad \psi(r) = a$$

whence follows that $\{\{a,b\},\{b,c\},\{c,r\},\{r,a\}\}$ is a meridian orbit. This, with (2.17), implies that $r = d$. Thus, if we let $\alpha \equiv \boxed{a,c;b,b}$, (2) holds.

[(2)$\Longrightarrow$(3)] Suppose now that (2) holds. By (2.18) there exists $\tau \in \llbracket \mathcal{M} \rrbracket_1$ such that (2.18.1) and (2.18.2) hold. It follows from (2.17) that $\tau(b) = c$. Thus (3) holds.

[(3)$\Longrightarrow$(1)] This follows from (2.18.1) and (2.18.2). Q.E.D.

The following figure illustrates harmony within the context of a circle meridian. The set $\{\{a,c\},\{b,d\}\}$ is a pair of harmonic pairs and $\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\}$ is a meridian orbit:

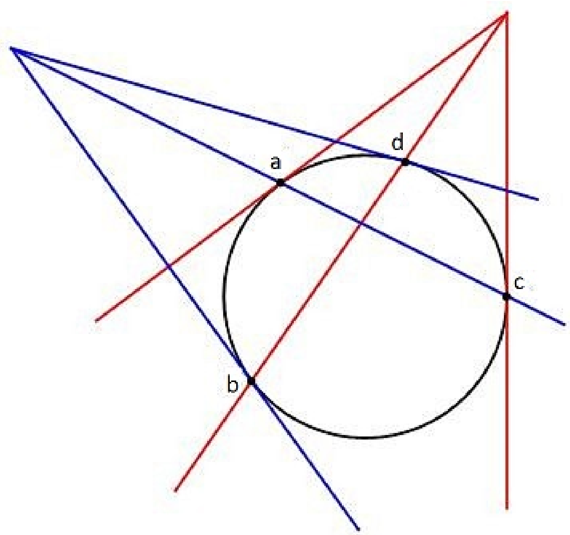


Fig. 6: Harmony on a Circle Meridian

(2.20) Mea Culpa So far as I know Jacques Tits was the first who was aware of the characterization of a meridian presented in the present section.

Since I have not done an exhaustive search of literature on the subject, I cannot claim originality for any of the contents contained in the present article . They merely represent my perception of the beauty of the subject entertained.

3. Erlanger Definition II

(3.1) Introduction Another way of expressing the fundamental theorem is to say that $[\![\mathcal{M}]\!]$ is **3-transitive** as a group acting on M. This begs the question: "how far does a 3-transitive group $\mathcal{G}$ acting on a set X go towards introducing a meridian structure on X?" It is the business of the present section to supply an answer, but first we formalize the meaning of 3-transitive.

(3.2) Definitions Let X be a set with cardinality at least 4. By a **basis for** X we shall mean a subset of X of cardinality 3.

A group $\mathcal{G}$ of permutations of X will be said to be **3-transitive** if, for each pair $\{\{a,b,c\},\{u,v,w\}\}$ of bases of X,

$$(\exists!\,\phi\in\mathcal{G})\quad \phi(a)=u,\ \phi(b)=v \text{ and } \phi(c)=w\,. \tag{1}$$

We shall find it helpful at times in the sequel to denote the function ϕ as

$$\boxed{\begin{smallmatrix} a & b & c\\ u & v & w\end{smallmatrix}}\,. \tag{2}$$

The **order** of an element ϕ of a group $\mathcal{G}$ of permutations is the least $n\in\mathbb{N}$ such that

$$\overbrace{\phi\circ\ldots\circ\phi}^{\text{n times}} \;=\; \iota_X\,. \tag{3}$$

For a group $\mathcal{G}$ of permutations, we introduce the notation

$$(\forall\, n\in\mathbb{N})\quad \mathcal{G}_n \;\equiv\; \{\phi\in\mathcal{G}:\phi \text{ has order } n\}\,. \tag{4}$$

(3.3) Definition We shall say that a 3-transitive group $\mathcal{G}$ of permutations of a set X is a **meridian group of permutations of** X if the following two additional conditions hold:

$$(\forall\,\{a,b,c\}\subset M\colon\ \#(\{a,b,c\})=3)(\exists!\,\phi\in\mathcal{G}_4)\quad \phi(a)=b,\quad \phi(b)=c\quad \text{and}\quad \phi^{-1}(a)=\phi(c) \tag{1}$$

and

$$(\forall\,\phi\in\mathcal{G}\colon\ (\exists\, x\in X)\ \phi\circ\phi(x)=x)\quad \phi\in\mathcal{G}_2\,. \tag{2}$$

We note that the statement $\phi^{-1}(a)=\phi(c)$ in (1) is superfluous, since it follows from the fact that ϕ has order 4. We include it because is provides clarity in some of the proofs *infra*.

(3.4) Theorem Let $\mathcal{M}$ be a meridian family of involutions of a set M. Then $[\![\mathcal{M}]\!]$ is a meridian group of permutations.

Proof. That (3.3.1) holds follows from (2.17.1). That (3.3.2) holds follows from Corollary (2.11). Q.E.D.

(3.5) Lemma Let $\mathcal{G}$ be a meridian group of permutations of a set X with at least 4 elements. Let p and q be distinct points of X. Then

$$(\exists!\,\phi\in\mathcal{G}_2)\quad \phi(p)=p\quad \text{and}\quad \phi(q)=q\,. \tag{1}$$

Proof. Let $x\in X$ be distinct from p and q. By (3.3.1) there exists a unique $\rho\in\mathcal{G}_4$ such that

$$\rho(p)=x,\quad \rho(x)=q\quad \text{and}\quad \rho(q)=\rho^{-1}(p)\,. \tag{2}$$

Let $y\equiv\rho(q)$. Direct calculation with (2) shows that

$$\boxed{\begin{smallmatrix}x&q&y\\p&y&q\end{smallmatrix}}\circ\rho(x)=y,\quad \boxed{\begin{smallmatrix}x&q&y\\p&y&q\end{smallmatrix}}\circ\rho(y)=x,\quad \boxed{\begin{smallmatrix}x&q&y\\p&y&q\end{smallmatrix}}\circ\rho(p)=p \quad\text{and}\quad \boxed{\begin{smallmatrix}x&q&y\\p&y&q\end{smallmatrix}}\circ\rho(q)=q,$$

which, along with (3.3.2) establishes the existence of $\phi\equiv\boxed{\begin{smallmatrix}x&q&y\\p&y&q\end{smallmatrix}}\circ\rho$ in $\mathcal{G}_2$.

Suppose that $\theta\in\mathcal{G}_2$ satisfies

$$\theta(p)=p \quad\text{and}\quad \theta(q)=q\,.$$

Let $z\equiv\theta(x)$. We have

$$\boxed{\begin{smallmatrix}p&q&z\\x&z&q\end{smallmatrix}}\circ\theta(p)=x,\quad\text{and}\quad \boxed{\begin{smallmatrix}p&q&z\\x&z&q\end{smallmatrix}}\circ\theta(x)=q$$

which by the uniqueness part of (3.3.2) implies that $\boxed{\begin{smallmatrix}x&q&z\\p&z&q\end{smallmatrix}}\circ\theta=\rho$ and $z=y$. Consequently

$$\theta(q)=\boxed{\begin{smallmatrix}x&q&z\\p&z&q\end{smallmatrix}}\circ\rho(q)=z=\phi(x)\,.$$

Since θ and ϕ agree at p and q as well, it follows that they are identical. Q.E.D.

(3.6) Lemma Let α and β be distinct elements of $\mathcal{G}_2$ which fix a common point p. Then

$$(\forall x\in M:\ \alpha\circ\beta(x)=x)\quad x=p\,. \tag{1}$$

Proof. Let x be as in (1) and assume that $x\neq p$. Then $\alpha(x)=\beta(x)$. Since α and β are distinct, it follows from (3.3.2) that $\alpha(x)\neq x$. Thus, if $y\equiv\alpha(x)$, then $\alpha(y)=\beta(y)$ and so $\alpha=\beta$. Q.E.D.

(3.7) Lemma Let $\mathcal{G}$ be a meridian group of permutations of a set X with at least 4 elements. Let p, x and y be distinct points of X. Then

$$(\exists!\,\tau\text{ a translation})\quad \tau(p)=p \quad\text{and}\quad \tau(x)=y\,. \tag{1}$$

Proof. From (3.5) follows that there exists $\alpha\in\mathcal{G}_2$ such that

$$\alpha(p)=p \quad\text{and}\quad \alpha(y)=y\,. \tag{2}$$

Let $\tau\equiv\alpha\circ\boxed{\begin{smallmatrix}p&x&y\\p&y&x\end{smallmatrix}}$. It follows from Lemma (6) that τ is a translation, which establishes the existence part of (1).

Let $z\equiv\tau(y)$. We assert that

$$(\exists\,\rho\in\mathcal{G}_4)\quad \rho(p)=z,\quad \rho(z)=y,\quad \rho(y)=x \quad\text{and}\quad \rho(x)=p. \tag{3}$$

First note that, in view of (3.3.2), both $\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}x&y&z\\y&x&p\end{smallmatrix}}$ are in $\mathcal{G}_2$. Since $\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}\circ\tau$ interchanges x and y, it also is an involution. Thus

$$p=\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}\circ\tau\circ\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}\circ\tau(p)=\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}\circ\tau\circ\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}(p) \implies \tau\Big(\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}(p)\Big)=\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}(p) \implies \boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}(p)=p\,.$$

Letting $\rho\equiv\boxed{\begin{smallmatrix}x&y&z\\y&x&p\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}$, direct calculation shows that (3) holds.

Now suppose that θ is another translation such that $\theta(p)=p$ and $\theta(x)=y$. Let $w\equiv\theta(y)$. Proceeding as above we obtain an element σ of $\mathcal{G}_2$ such that

$$\sigma(p)=w,\quad \sigma(w)=y,\quad \sigma(y)=x \quad\text{and}\quad \sigma(x)=p. \tag{4}$$

From (3) follows

$$\rho(y)=x,\quad \rho(x)=p \quad\text{and}\quad \rho(p)=\rho^{-1}(y)$$

and from (4) follows

$$\sigma(y)=x,\quad \sigma(x)=p \quad\text{and}\quad \sigma(p)=\sigma^{-1}(y)\,.$$

From (3.3.1) follows that $\sigma=\rho$, which implies that $w=z$. Thus τ and σ agree on three points, and so are equal. Q.E.D.

(3.8) Theorem Let $\mathcal{G}$ be a meridian group of permutations of a set X with at least 4 elements. Then $\mathcal{G}_2$ is a meridian family of permutations of X and $\mathcal{G}=\llbracket\mathcal{G}_2\rrbracket$.

Proof. That (2.3.1) holds is evident, so we proceed to establishing (2.3.2). Let $\{a,b,c,d\}\subset X$ be such that $a\neq b\neq c\neq d\neq a$. We consider two separate cases.

[Case 1: $a=c$ and $b=d$] That there exists a unique $\phi\in\mathcal{G}_2$ such that (2.3.2) holds in this case is a consequence of (3.5).

[Case 2: $a\neq c$ or $b\neq d$] If $a\neq c$ we let $\phi\equiv\boxed{\begin{smallmatrix}a&b&c\\c&d&a\end{smallmatrix}}$ — otherwise we let $\phi\equiv\boxed{\begin{smallmatrix}a&b&d\\c&d&b\end{smallmatrix}}$. It follows from (3.3.2) that ϕ is an involution. That the equation in (2.3.2) holds is now trivial.

Now we shall establish (2.3.3). Let a and b be in X and let α, β and γ be elements of $\mathcal{G}_2$, each of which sends a to b. Again we shall treat two separate cases.

[Case 1: $a\neq b$] That both $\alpha\circ\beta\circ\gamma(a)=b$ and $\alpha\circ\beta\circ\gamma(b)=a$ is evident. It follows from (3.3.2) that $\alpha\circ\beta\circ\gamma(b)$ is an involution.

[Case 2: $a=b$] That $\alpha\circ\beta\circ\gamma(a)=b$ is trivial. We must show that $\alpha\circ\beta\circ\gamma$ is an involution. If $\alpha=\beta$, this is trivial so we may and shall presume that $\alpha\neq\beta$. Towards our purpose we let x be any element of X distinct from a an define

$$y\equiv\alpha\circ\beta(x),\quad z\equiv\gamma(x)\quad\text{ and }\quad\delta\equiv\boxed{\begin{smallmatrix}a&y&z\\a&z&y\end{smallmatrix}}.$$

By (3.6) we have $y\neq x$. Thus $\delta\neq\gamma$. That $\alpha\circ\beta$ and $\delta\circ\gamma$ are in $\mathcal{G}_1$ follows from Lemma (3.6). We have

$$\alpha\circ\beta(a)=a=\delta\circ\gamma(a)\quad\text{ and }\quad\alpha\circ\beta(x)=y=\delta\circ\gamma(x).$$

It follows from (3.7) that

$$\alpha\circ\beta=\delta\circ\gamma.$$

Consequently $\alpha\circ\beta\circ\gamma=\delta$, which is an involution.

It remains to verify that that $\mathcal{G}\subset\llbracket\mathcal{G}_2\rrbracket$. Let ϕ be in $\mathcal{G}$. We may and shall presume that $\phi\neq\iota_X$. Let w be any element of X such that $\phi(w)\neq w$. Let $x\equiv\phi(w)$, $y\equiv\phi(x)$ and $z\equiv\phi(y)$.

[Case 1: $y=w$] It follows from (3.3.2) that ϕ is in $\mathcal{G}_2$.

[Case 2: $y\neq w$ and $z=w$] Then $\phi\circ\boxed{\begin{smallmatrix}w&x&y\\w&y&x\end{smallmatrix}}$ sends w to x and x to w which, by (3.3.2), implies that $\alpha\equiv\phi\circ\boxed{\begin{smallmatrix}w&x&y\\w&y&x\end{smallmatrix}}$ is in $\mathcal{G}_2$. Consequently $\phi=\alpha\circ\boxed{\begin{smallmatrix}w&x&y\\w&y&x\end{smallmatrix}}$, which is in $\llbracket\mathcal{G}_2\rrbracket$.

[Case 3: $\#(\{w,x,y,z\})=4$] Then ϕ agrees with $\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}x&y&z\\y&x&w\end{smallmatrix}}$ on w, x, and y. Thus $\phi=\boxed{\begin{smallmatrix}x&y&z\\z&y&x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}x&y&x\\y&z&w\end{smallmatrix}}$, and so is in $\llbracket\mathcal{G}_2\rrbracket$. Q.E.D.

(3.9) Remark It is a consequence of Theorems (3.4) and (3.8) that a meridian could have been defined as a set X of at least cardinality 4 having a meridian family of permutations. This choice would have been more strictly consistent with the Erlanger Programm, which requires a group of permutations.

(3.10) Terminology Let $\mathcal{G}$ be a meridian group of permutations on a set M with at least cardinality 4. We have already appropriated the term **involution** for a self-inverse element of $\mathcal{G}$. The general term for any element of $\mathcal{G}$ with two fixed points is **dilation**. An element of $\mathcal{G}$ with one fixed point is called a **translation**. We shall call elements of $\mathcal{G}$ which are neither involutions, dilations nor translations, **pure rotations**. The following characterizations follow from the results of the present section: for $\phi\in\mathcal{G}$ such that $\phi\neq\phi^{-1}$,

$$\phi\text{ is a pure rotation}\iff(\exists\,\{\alpha,\beta\}\subset\mathcal{G}_2)\quad\phi=\alpha\circ\beta,\quad\alpha\circ\beta\neq\beta\circ\alpha\quad\text{ and }\quad(\forall\,m\in M)\ \beta(m)\neq\alpha(m)\,,\quad(1)$$

$$\phi \text{ is a translation} \iff (\exists\, \{\alpha,\beta\} \subset \mathcal{G}_2)(\exists!\, m \in M) \quad \phi = \alpha\circ\beta, \quad \alpha\circ\beta \neq \beta\circ\alpha \quad \text{and} \quad \alpha(m) = \beta(m) = m \tag{2}$$

and

$$\phi \text{ is a dilation} \iff (\exists\, m \in M \text{ and } \{\alpha,\beta\} \subset \mathcal{G}_2 \text{ distinct}) \quad \phi = \alpha\circ\beta, \quad \alpha\circ\beta \neq \beta\circ\alpha \quad \text{and} \quad \alpha(m) = \beta(m) \neq m\,. \tag{3}$$

Proof. [(1) holds] Let f be a pure rotation. By (3.3.2) there exists $\{\alpha,\beta\} \subset \mathcal{G}_2$ such that $\phi = \alpha\circ\beta$. If β and α agreed at any point $m \in M$, then ϕ would have a fixed point. If α and β commuted with one another, then ϕ would be an involution.

If the right side of (1) holds, than it is trivial that ϕ has no fixed point.

[(2) holds] Let f be a translation. By (3.8) we know that M is a meridian. Let m be the fixed point of ϕ, let $n \in M$ be distinct from m, let $a \equiv \phi^{-1}(n)$ and let $b \equiv \phi(n)$. It follows from (2.19.3) that $\{\{m,n\},\{a,b\}\}$ is an harmonic pair and so (2.19.2) implies that there exists $\beta \in \mathcal{M}$ such that $\beta(a) = b$, $\beta(m) = n$ and $\beta(n) = n$. Defining $\alpha \equiv \phi\circ\beta$, we have

$$\alpha(n) = \phi\circ\beta(n) = \phi(n) = b \quad \text{and} \quad \alpha(b) = \phi\circ\beta(b) = \phi(a) = n \quad \xRightarrow{\text{by (3.3.2)}} \quad \alpha \in \mathcal{M}$$

Evidently $\alpha(m) = m$ and, since ϕ is not an involution, we have $\alpha\circ\beta \neq \beta\circ\alpha$.

Now suppose that $\phi = \alpha\circ\beta$ for $\{\alpha,\beta\} \subset \mathcal{G}_2$, that $\alpha\circ\beta \neq \beta\circ\alpha$ and that $\alpha(m) = \beta(m) = m$. If $\phi(n) = n$ for $n \neq m$, then α would equal β at n as well as m and so by the fundamental theorem would be equal. Hence ϕ is a translation.

[(3) holds] Let ϕ be a non-involutive dilation, let 0 and ∞ be the distinct fixed points of ϕ and let 1 be any point in M distinct from 0 and ∞ .Let $\alpha \equiv \boxed{0,\infty;1,\phi(1)}$ and $\beta \equiv \boxed{0,\infty;1,1}$. Then

$$\phi = \boxed{\begin{smallmatrix} 0 & 1 & \infty \\ 0 & 1 & \phi(1) \end{smallmatrix}} = \boxed{0,\infty;1,\phi(1)} \circ \boxed{0,\infty;1,1} = \alpha\circ\beta\,, \; \alpha(0) = \infty = \beta(0)$$

and $\alpha\circ\beta \neq \beta\circ\alpha$ since ϕ is not an involution.

Finally, let $\phi = \alpha\circ\beta$ for $\{\alpha,\beta\} \subset \mathcal{G}_2$, $\alpha\circ\beta \neq \beta\circ\alpha$ and $\alpha(m) = \beta(m) \neq m$ for some $m \in M$. Then

$$\phi(m) = m \neq \alpha(m) = \phi(\alpha(m))$$

and ϕ is not an involution since α an β do not commute. Q.E.D.

4. Involution Libras

(4.1) Introduction Our next characterization of a meridian will be in terms of "balanced" functions on the faces of a cube. An important part of this characterization involves a simpler sort of object which we shall term a "libra". The basic notion behind it is a set of scales – hence the name. For brevity however, we shall take a short cut past the scales, leaving those for later.

(4.2) Definitions Let L be a set and $\lfloor,,\rfloor \,\big|\, L\times L\times L \hookrightarrow L$ a trinary operator on L for which the following holds:

$$(\forall\,\{a,b\}\subset L)\quad \lfloor a,a,b\rfloor = b = \lfloor b,a,a\rfloor \tag{1}$$

and

$$(\forall\,\{a,b,c,d,e\}\subset L)\quad \lfloor\lfloor a,b,c\rfloor,d,e\rfloor = \lfloor a,b,\lfloor c,d,e\rfloor\rfloor\,. \tag{2}$$

Then $\lfloor,,\rfloor$ will be said to be a **libra operator** and L, relative to $\lfloor,,\rfloor$ a **libra**. A subset B of a libra will be said to be **balanced** provided $\lfloor a,b,c\rfloor$ is in B whenever $\{a,b,c\}\subset B$.

(4.3) Theorem Let $\lfloor,,\rfloor$ be a libra operator on a set L. Then

$$(\forall\,\{a,b,c,e,f\}\subset L)\quad \lfloor a,\lfloor d,c,b\rfloor,e\rfloor = \lfloor\lfloor a,b,c\rfloor,d,e\rfloor\,. \tag{1}$$

Proof. We have

$$\begin{aligned} a &\overset{\text{by (4.2.1)}}{=\!=} \lfloor a,b,b\rfloor \overset{\text{by (4.2.1)}}{=\!=} \lfloor a,b,\lfloor c,c,b\rfloor\rfloor \overset{\text{by (4.2.1)}}{=\!=} \lfloor a,b\lfloor\lfloor c,d,d\rfloor,c,b\rfloor\rfloor \overset{\text{by (4.2.2)}}{=\!=} \\ &\lfloor\lfloor a,b,\lfloor c,d,d\rfloor\rfloor,c,b\rfloor \overset{\text{by (4.2.2)}}{=\!=} \lfloor\lfloor\lfloor a,b,c\rfloor,d,d\rfloor,c,b\rfloor \overset{\text{by (4.2.2)}}{=\!=} \lfloor\lfloor a,b,c\rfloor,d\lfloor d,c,b\rfloor\rfloor \end{aligned} \tag{2}$$

whence follows

$$\begin{aligned} &\lfloor a,\lfloor d,c,b\rfloor,e\rfloor \overset{\text{by (2)}}{=\!=} \lfloor\lfloor\lfloor a,b,c\rfloor,d,\lfloor d,c,b\rfloor\rfloor,\lfloor d,c,b\rfloor,e\rfloor \overset{\text{by (4.2.2)}}{=\!=} \\ &\lfloor\lfloor a,b,c\rfloor,d,\lfloor\lfloor d,c,b\rfloor,\lfloor d,c,b\rfloor,e\rfloor\rfloor \overset{\text{by (4.2.1)}}{=\!=} \lfloor\lfloor a,b,c\rfloor,d,e\rfloor\,. \end{aligned}$$

Q.E.D.

(4.4) Convention The various compositions of libra operators with libra operators, in view of (4.2.1), (4.2.2) and (4.3.1), may be greatly simplified: we define

$$\lfloor a,b,c,d,e\rfloor \;\equiv\; \lfloor\lfloor a,b,c\rfloor,d,e\rfloor \;=\; \lfloor a,\lfloor d,c,b\rfloor,e\rfloor \;=\; \lfloor a,b,\lfloor c,d,e\rfloor\rfloor\,. \tag{1}$$

Each such composition may be converted to a form

$$\lfloor a_1,a_2,\lfloor a_3,a_4,\lfloor\ldots\lfloor a_{n-2},a_{n-1},a_n\rfloor\ldots\rfloor\rfloor\rfloor \tag{2}$$

for n an odd positive integer. We shall at times adopt the abbreviation

$$\lfloor a_1,a_2,\ldots,a_n\rfloor \tag{3}$$

for (2).

(4.5) Example Let A be an affine space over a field F. Then the **translations** of A form a vector space over F. The translation of a point a∈A by a vector v∈V is denoted by v+a. To any two distinct points a and b in A corresponds a unique vector (which we denote by b-a) such that (b-a)+a = b. Then

$$(\forall\, \{a,b,c\} \subset A) \quad \lfloor a,b,c \rfloor \equiv (a\text{-}b)+c \tag{1}$$

defines a libra operator. We have $d = \lfloor a,b,c \rfloor$ precisely when the points a,b, c and d describe the points of a parallelogram.[23]

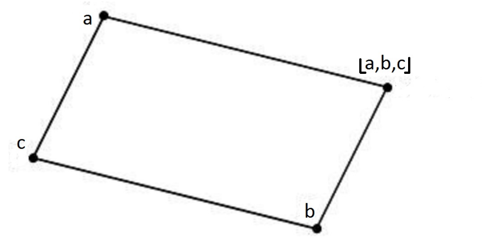


Fig. 7: Affine Libra Operator

(4.6) Example We have already seen an important example in (2.2): the libra of functions.

(4.7) Theorem Let $\lfloor ,, \rfloor$ be a libra operator on a libra L and e be an element of L. Then the binary operation

$$\cdot \Big| L \times L \ni [x,y] \hookrightarrow x \cdot y \equiv \lfloor x,e,y \rfloor \in L \tag{1}$$

is a group operation on L, relative to which e is the identity and

$$(\forall\, x \in L) \quad \lfloor e,x,e \rfloor \text{ is the group inverse of e.} \tag{2}$$

Proof. For $\{x,y,z\} \subset L$

$$(x \cdot y) \cdot z = \lfloor \lfloor x,e,y \rfloor ,e,z \rfloor \overset{\text{by } (4.2.2)}{=\!=\!=} \lfloor x,e \lfloor y,e,z \rfloor \rfloor = x \cdot (y \cdot z),$$

$$x \cdot e = \lfloor x,e,e \rfloor \overset{\text{by } (4.2.1)}{=\!=\!=} x \overset{\text{by } (4.2.1)}{=\!=\!=} \lfloor e,e,x \rfloor = e \cdot x,$$

$$x \cdot \lfloor e,x,e \rfloor = \lfloor x,e, \lfloor e,x,e \rfloor \rfloor \overset{\text{by } (4.2.2)}{=\!=\!=} \lfloor \lfloor x,e,e \rfloor ,x,e \rfloor \overset{\text{by } (4.2.1)}{=\!=\!=} \lfloor x,x,e \rfloor \overset{\text{by } (4.2.1)}{=\!=\!=} e$$

and $$\lfloor e,x,e \rfloor \cdot x = \lfloor \lfloor e,x,e \rfloor ,e,x \rfloor \overset{\text{by } (4.2.2)}{=\!=\!=} \lfloor e,x, \lfloor e,e,x \rfloor \rfloor \overset{\text{by } (4.2.1)}{=\!=\!=} \lfloor e,x,x \rfloor \overset{\text{by } (4.2.1)}{=\!=\!=} e\,.$$

(4.8) Theorem Let G be a group with binary operation $\cdot$. Define the trinary operator

$$\lfloor ,, \rfloor \Big| G \times G \times G \ni [a,b,c] \hookrightarrow a \cdot b^{-1} \cdot c \in G\,. \tag{1}$$

Then $\lfloor ,, \rfloor$ is a libra operator.

Proof. For $\{r,s,t,t,v\} \subset G$,

$$\lfloor r,s,s \rfloor = r \cdot s^{-1} \cdot s = r = s \cdot s^{-1} \cdot r = \lfloor s,s,r \rfloor$$

and $$\lfloor \lfloor r,s,t \rfloor ,u,v \rfloor = (r \cdot s^{-1} \cdot t) \cdot u^{-1} \cdot v = r \cdot s^{-1} \cdot (t \cdot u^{-1} \cdot v) = \lfloor r,s, \lfloor t,u,v \rfloor \rfloor\,.$$

Q.E.D.

(4.9) Definition The libra operator defined in (4.8) will be called the **group libra operator**.

[23] taken in clockwise, or counter-clockwise order.

(4.10) Definitions A function ϕ from one libra L_1 into another L_2 which preserves the libra operator is called a **libra homomorphism**. Thus a libra homomorphism ϕ is characterized by

$$(\forall \{a,b,c\}\subset L_1) \quad \lfloor \phi(a),\phi(b),\phi(c)\rfloor = \phi(\lfloor a,b,c\rfloor) . \tag{1}$$

A bijective libra homomorphism is a **libra isomorphism**.

(4.11) Theorem Let G and H be two groups, and let ϕ be a group homomorphism from G into H. Then ϕ is also a libra homomorphism.

Proof. For $\{a,b,c\}\subset G$ we have

$$\lfloor \phi(a),\phi(b),\phi(c)\rfloor = \phi(a)\cdot(\phi(b))^{-1}\cdot\phi(d) = \phi(a\cdot b^{-1}\cdot c) = \phi(\lfloor a,b,c\rfloor) .$$

(4.12) Definitions and Notation A libra L will be called **abelian** if

$$\lfloor a,b,c\rfloor = \lfloor c,b,a\rfloor \text{ for all } \{a,b,c\}\subset L. \tag{1}$$

Evidently L is abelian if and only if each of its corresponding groups is abelian.

For a and b in a libra L, we define the functions

$${}_a\pi_b\big| L\ni x\hookrightarrow \lfloor a,x,b\rfloor\in L, \quad {}_a\lambda_b\big| L\ni x\hookrightarrow \lfloor a,b,x\rfloor\in L \quad \text{and} \quad {}_a\rho_b\big| L\ni x\hookrightarrow \lfloor x,a,b\rfloor\in L . \tag{2}$$

The functions ${}_a\rho_b$ and ${}_a\lambda_b$, respectively, are called **libra right translations** and **libra left translations**, respectively. When L is an abelian libra, the function ${}_a\pi_b$ will be called a **libra inner involution**.

(4.13) Theorem Let $\lfloor ,, \rfloor$ be an abelian libra operation on a set L. Let $\Pi(L)$ denote the set of inner involutions on L. Then

$$(\forall \phi\in\Pi(L)) \quad \phi=\phi^{-1} , \tag{1}$$

$$(\forall \{a,b\}\subset L)(\exists!\, \phi\in\Pi(L)) \quad \phi(a) = b \tag{2}$$

and

$$(\forall \{\alpha,\beta,\gamma\}\subset\Pi(L)) \quad \alpha\circ\beta\circ\gamma\in\Pi(L) . \tag{3}$$

Proof. For $\{r,s,t,u,v,w,x\}\subset L$

$${}_r\pi_s\circ{}_r\pi_s(x) = \lfloor r,\lfloor r,x,s\rfloor,s\rfloor = \lfloor r,\lfloor s,x,r\rfloor,s\rfloor \overset{\text{by (4.3.1)}}{=\!=\!=} \lfloor r,r,x,s,s\rfloor \overset{\text{by (4.2.1)}}{=\!=\!=} x ,$$

$${}_r\pi_s = \lfloor r,r,s\rfloor \overset{\text{by (4.2.1)}}{=\!=\!=} s ,$$

and, if we let $a\equiv\lfloor r,u,v\rfloor$ and $b\equiv\lfloor w,t,s\rfloor$,

$${}_r\pi_s\circ{}_t\pi_u\circ{}_u\pi_w(x) = \lfloor r,\lfloor t,\lfloor v,x,w\rfloor,u\rfloor,s\rfloor \overset{\text{by (4.3.1)}}{=\!=\!=} \lfloor r,\lfloor\lfloor t,w,\lfloor x,v,u\rfloor\rfloor\rfloor,s\rfloor \overset{\text{by (4.3.1)}}{=\!=\!=}$$
$$\lfloor r,\lfloor x,v,u\rfloor,\lfloor w,t,s\rfloor\rfloor = \lfloor r,u,v,x,w,t,s\rfloor = \lfloor\lfloor r,u,v\rfloor,x,\lfloor w,t,s\rfloor\rfloor = {}_a\pi_b$$

It remains only to show that ${}_r\pi_s$ is the only element of $\Pi(L)$ which sends r to s. Suppose that ${}_t\pi_u$ is another such. Then $s=\lfloor t,r,u\rfloor$, whence, for each $x\in L$,

$${}_r\pi_s(x) = \lfloor r,x,s\rfloor = \lfloor r,x\lfloor t,r,u\rfloor\rfloor \overset{\text{by (4.3.1)}}{=\!=\!=} \lfloor r,\lfloor r,t,x\rfloor,u\rfloor =$$
$$\lfloor r,\lfloor x,t,r\rfloor,u\rfloor \overset{\text{by (4.3.1)}}{=\!=\!=} \lfloor\lfloor r,r,t\rfloor,x,u\rfloor \overset{\text{by (4.2.1)}}{=\!=\!=} \lfloor t,x,u\rfloor = {}_t\pi_u(x) .$$

Q.E.D.

(4.14) Definition A family $\varPi$ of operators of a set S satisfying conditions (4.13.1), (4.13.2) and (4.13.3) of (4.13) will be said to be an **inner involution libra on** S.

(4.15) Example and Theorem Let $\mathcal{M}$ be a meridian involution family on a meridian M. For $\{a,b\}\subset M$, the family

$$\overset{a\leftrightarrow b}{\mathcal{M}} \equiv \{\phi\in\mathcal{M} : \phi(a)=b\} \tag{1}$$

is an abelian function libra by definition (2.3.3). By (2.3.2) and by (4.14), $\overset{a\leftrightarrow b}{\mathcal{M}}$ is an inner involution libra on

$$M_{(a,b)} \equiv M\triangle\{a,b\}\,. \tag{2}$$

Let $\{a,b,c,d\}\subset M$. Then

$$\text{if } a\neq b \text{ and } c\neq d\text{, then } \overset{a\leftrightarrow b}{\mathcal{M}} \text{ and } \overset{c\leftrightarrow d}{\mathcal{M}} \text{ are isomorphic as libras,} \tag{3}$$

$$\overset{a\leftrightarrow a}{\mathcal{M}} \text{ and } \overset{c\leftrightarrow c}{\mathcal{M}} \text{ are isomorphic as libras} \tag{4}$$

and

$$\text{if } a\neq b\text{, then } \overset{a\leftrightarrow b}{\mathcal{M}} \text{ and } \overset{c\leftrightarrow c}{\mathcal{M}} \text{ are not isomorphic as libras.} \tag{5}$$

Proof. Suppose first that $a\neq b$ and $c\neq d$. By the Fundamental Theorem there exists $\psi\in[\![\mathcal{M}]\!]$ such that

$$\psi(a)=c \quad\text{and}\quad \psi(b)=d\,.$$

The function

$$\overset{a\leftrightarrow b}{\mathcal{M}}\ni\theta\hookrightarrow\psi^{-1}\circ\theta\circ\psi\in\overset{c\leftrightarrow d}{\mathcal{M}} \tag{6}$$

is a libra isomorphism.

Now suppose that $a=b$ and that $c=d$. By the Fundamental Theorem there exists $\psi\in[\![\mathcal{M}]\!]$ such that

$$\psi(a)=c\,.$$

Again, the function in (6) is a libra isomorphism.

Finally we suppose that $a\neq b$ and assume that there existed a libra isomorphism $\mathfrak{w}$ from $\overset{a\leftrightarrow b}{\mathcal{M}}$ onto $M_{(c,c)}$. Let $t\in M$ be distinct from both a and b. Since $\boxed{a,b;t,t}$ fixes t and is an involution, it follows from Theorem (2.15) that there would be exactly one other point $u\in M$ such that $\boxed{a,b;t,t}(u)=u$. From (2.3.1) follows that $\boxed{a,b;t,u}\circ\boxed{a,b;t,t}\circ\boxed{a,b;t,u}$ is in $\overset{a\leftrightarrow b}{\mathcal{M}}$. Furthermore

$$\boxed{a,b;t,u}\circ\boxed{a,b;t,t}\circ\boxed{a,b;t,u} \text{ is in } \overset{a\leftrightarrow b}{\mathcal{M}} \text{ and } \boxed{a,b;t,u}\circ\boxed{a,b;t,t}\circ\boxed{a,b;t,u}(u)=\boxed{a,b;t,u}\circ\boxed{a,b;t,t}(t)=\boxed{a,b;t,u}(t)=u\,.$$

Since $\boxed{a,b;t,t}$ and $\boxed{a,b;t,u}\circ\boxed{a,b;t,t}\circ\boxed{a,b;t,u}$ would agree on both t and u, it follows from the uniqueness part of (2.3.2) that

$$\boxed{a,b;t,t}=\boxed{a,b;t,u}\circ\boxed{a,b;t,t}\circ\boxed{a,b;t,u}\,. \tag{7}$$

Let α and β be the elements of $\overset{c\leftrightarrow c}{\mathcal{M}}$ such that $\mathfrak{w}(\alpha)=\boxed{a,b;t,t}$ and $\mathfrak{w}(\beta)=\boxed{a,b;t,u}$. From (7) follows that

$$\mathfrak{w}(\beta)\circ\mathfrak{w}(\alpha)\circ\mathfrak{w}(\alpha)=\mathfrak{w}(\beta) \quad\Longrightarrow\quad \beta\circ\alpha\circ\beta=\alpha\,.$$

Since β would fix c, from (2.15) follows that it would have another distinct fixed point p. Then

$$\alpha(p)=\beta\circ\alpha\circ\beta(p)=\beta(\alpha(p)) \quad\Longrightarrow\quad \alpha(p)\in\{c,p\}\,.$$

But $\alpha(c)=c$, and so $\alpha(p)=p$. It follows from the uniqueness part of (2.3.2) that $\alpha=\beta$. This however is absurd because $\boxed{a,b;t,t}$ and $\boxed{a,b;t,u}$ are distinct. Q.E.D.

(4.16) Example Let C be a circle as in (2.6). Let L be a line in the projective plane $\mathbb{P}$. For each $p\in L$, the function p_C is defined as in (2.6). The family $\{p_C : p\in L\}$ is an inner involution libra on C.

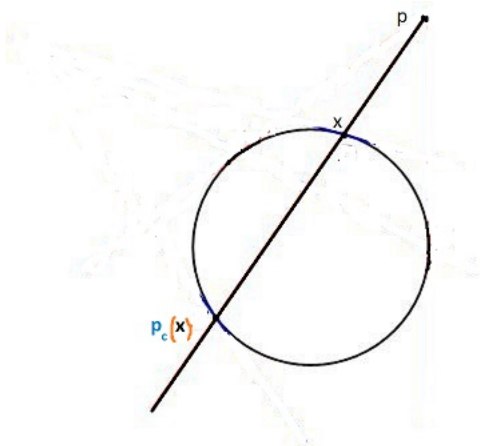


Fig. 8: Circle Meridian Involution

If the line L intersects C, either at two points a and b, or at a point of tangency c, then then the corresponding inner involution meridian is either the family $\overset{a\leftrightarrow b}{\mathcal{M}}$, or the family $\overset{c\leftrightarrow c}{\mathcal{M}}$ of (4.15).

(4.17) Corollary Relative to the trinary operator

$$\Pi(L)\times\Pi(L)\times\Pi(L)\ni[\alpha,\beta,\gamma]\hookrightarrow\alpha\circ\beta\circ\gamma\in\Pi(L)\,,$$

$\Pi(L)$ is a function libra.

(4.18) Theorem Let Π be a family of permutations of a set S such that

$$(\forall\,\phi\in\Pi)\quad \phi=\phi^{-1}\,, \tag{1}$$

$$(\forall\,\{a,b\}\subset S)(\exists!\,\phi\in\Pi)\quad \phi(a)=b\,, \tag{2}$$

and

$$(\forall\,\{\alpha,\beta,\gamma\}\subset\Pi)\quad \alpha\circ\beta\circ\gamma\in\Pi\,. \tag{3}$$

Let $\Theta\equiv\{\alpha\circ\beta : \{\alpha,\beta\}\subset\Pi\}$. Then

$$(\forall\,\{\alpha,\beta,\gamma,\delta\}\subset\Pi:\ (\exists\,s\in S)\quad \alpha\circ\beta(s)=\gamma\circ\delta(s))\quad \alpha\circ\beta=\gamma\circ\delta\,, \tag{4}$$

$$(\forall\,\{a,b,c\}\subset S)(\exists!\,{}_a\theta_b\in\Theta)\quad {}_a\theta_b(a)=b \tag{5}$$

and

$$\Theta \text{ is an abelian group under composition.} \tag{6}$$

Proof. If $\alpha\circ\beta(s)=\gamma\circ\delta(s)$, then $\gamma\circ\alpha\circ\beta(s)=\delta(s)$ and so (3) and (2) imply that $\gamma\circ\alpha\circ\beta=\delta$. This (4) holds.

Let α be the function in Π which leaves a fixed and β be the one which sends a to b. Then $\beta\circ\alpha(a)=b$. That $\beta\circ\alpha$ is unique with this property follows from (4), which proves (5).

For $\{\alpha,\beta,\gamma,\delta\}\subset\Pi$ we have $(\alpha\circ\beta)\circ(\gamma\circ\delta)=(\alpha\circ\beta\circ\gamma)\circ\delta$ which is in Θ by (3). That $\alpha\circ\alpha=\iota_S$ follows from (1). For $\{\alpha,\beta\}\subset\Pi$, we have $(\alpha\circ\beta)\circ(\beta\circ\alpha)=\iota_S$ by (1). Thus Θ is a group.

For $\{\alpha,\beta,\gamma,\delta\}\subset\Pi$ we have $(\alpha\circ\beta\circ\gamma)^{-1}=\gamma\circ\beta\circ\alpha$. By (1) and (2) this implies that

$$\alpha\circ\beta\circ\gamma=\gamma\circ\beta\circ\alpha\,.$$

Consequently

$$(\alpha\circ\beta)\circ(\gamma\circ\delta)=(\alpha\circ\beta\circ\gamma)\circ\delta=(\gamma\circ\beta\circ\alpha)\circ\delta=\gamma\circ(\beta\circ\alpha\circ\delta)=\gamma\circ(\delta\circ\alpha\circ\beta)=(\gamma\circ\delta)\circ(\alpha\circ\beta)$$

which proves (6). Q.E.D.

(4.19) Theorem Let Π be an inner involution libra on a set S. For all $\{a,c\}\subset S$ we shall denote by ${}_a\phi_c$ the function in Π which sends a to c and define

$$(\forall\,\{a,b,c\}\subset S)\quad \lfloor a,b,c\rfloor \equiv {}_a\phi_c(b) \tag{1}$$

Then $\lfloor,,\rfloor$ is a libra operator on S and, for each $x\in S$,

$$\Pi\ni\pi\hookrightarrow\pi(x)\in S \text{ is a libra isomorphism.} \tag{2}$$

Proof. Let a, b and c be generic elements of S. We have

$$\lfloor a,a,b\rfloor={}_a\phi_b(a)=b \quad\text{and}\quad \lfloor a,b,b\rfloor={}_a\phi_b(b)=a$$

by definition, which is just (4.2.1).

Let a, b, c, d and e be generic elements of S and let Θ be as in (4.18). For all $\{x,y\}\subset S$, let ${}_x\theta_y$ be as in (4.18.5). It follows from (4.18) that

$$(\forall\,x\in S)\quad {}_a\phi_x\circ{}_b\phi_x(b)=a \implies (\forall\,x\in S)\quad {}_a\phi_x\circ{}_b\phi_x(b)={}_b\theta_a(b) \tag{3}$$

and

$$(\forall\,x\in S)\quad {}_x\phi_e\circ{}_x\phi_d(b)=e \implies (\forall\,x\in S)\quad {}_x\phi_e\circ{}_x\phi_d(b)={}_d\theta_e(b)\,. \tag{4}$$

Letting $u\equiv{}_b\theta_a(c)$ and $v\equiv{}_d\theta_e(c)$, we have

$$\lfloor\lfloor a,b,c\rfloor,d,e\rfloor=\lfloor{}_a\phi_c(b),d,e\rfloor=\lfloor{}_a\phi_c\circ{}_b\phi_c(c),d,e\rfloor\overset{\text{by (3)}}{=\!=}\lfloor{}_b\theta_a(c),d,e\rfloor=\lfloor u,d,e\rfloor=$$

$${}_a\phi_e(d)={}_u\phi_e\circ{}_u\phi_d(u)\overset{\text{by(4)}}{=\!=}{}_d\theta_e={}_d\theta_e\circ{}_b\theta_a(c)\overset{\text{by (4.18.6)}}{=\!=}{}_b\theta_a\circ{}_d\theta_e(c)={}_b\theta_a(v)\overset{\text{by (3)}}{=\!=}{}_a\phi_v\circ{}_b\phi_v(v)=$$

$${}_a\phi_v(b)=\lfloor a,b,v\rfloor=\lfloor a,b,{}_d\theta_e\rfloor\overset{\text{by (4)}}{=\!=}\lfloor a,b,{}_c\phi_e\circ{}_c\phi_d(c)\rfloor=\lfloor a,b,{}_c\phi_e(d)\rfloor=\lfloor a,b,\lfloor c,d,e\rfloor\rfloor$$

which establishes that $\lfloor,,\rfloor$ is a libra operator.

Let x be in S and α, β and γ be in Π. Let $c\equiv\gamma(x)$, $b\equiv\beta(c)$ and $a\equiv\alpha(b)$. Then

$$\alpha={}_a\phi_b\,,\quad \beta={}_b\phi_c\quad\text{and}\quad \gamma={}_c\phi_x\,.$$

We have

$$\llbracket\alpha,\beta,\gamma\rrbracket(x)=\alpha\circ\beta\circ\gamma(x)={}_a\phi_b\circ{}_b\phi_c\circ\gamma(x)={}_a\phi_b\circ{}_b\phi_c\circ{}_c\phi_x(x)=a=\lfloor a,x,b,b,x,c,c,x,x\rfloor=$$

$$\lfloor\lfloor a,x,b\rfloor,\lfloor b,x,c\rfloor,\lfloor c,x,x\rfloor\rfloor=\lfloor{}_a\phi_b(x),{}_b\phi_c(x),{}_c\phi_x(x)\rfloor=\lfloor\alpha(x),\beta(x),\gamma(x)\rfloor$$

which establishes (4.2.2) Q.E.D.

(4.20) Example Let C be the circle libra in the context of (2.6). Let L be a line in the projective plane $\mathbb{P}$. As in example (4.16), the points p of L correspond to involutions p_C of C in such a way that $\{p_C : p\in L\}$ comprise an inner involution libra. This induces a libra on the points of L. The first figure below illustrates the induced libra on the line and how computation of the libra operation is independent of the choice of point in C used to compute it:

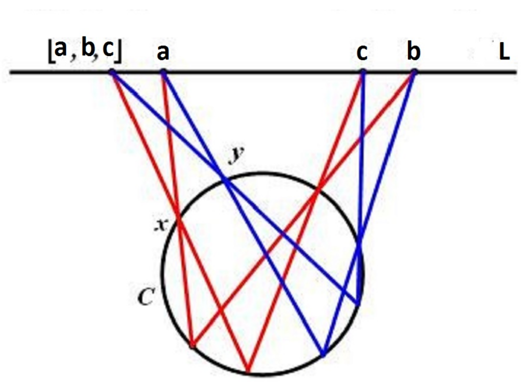


Fig. 9: Libra on the Line Induced by a Circle

In view of Theorem (4.19), an isomorphic libra is induced on C as well. The next figure illustrates this libra operation on C:

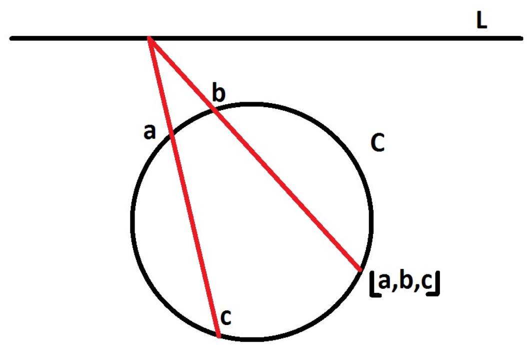
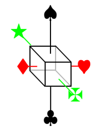


Fig. 10: Libra on the Circle Induced by a Line

5. Meridians on the Cube

(5.1) Introduction Consider a cube



.

We shall write $\mathcal{F}$ for the set of faces of the cube. There are six faces ✠, ★, ♠, ♣, ♥ and ♦.

We shall denote by $\mathbb{K}$ the family of permutations of the faces of the cube which preserve the cube's physical structure intact.

In addition we shall require other permutations of $\mathcal{F}$ which are not in $\mathbb{K}$. We shall write the permutation of $\mathcal{F}$ which interchanges the green faces and leaves the others fixed by $\pi_{\color{green}\blacksquare}$. The involutions $\pi_{\blacksquare}$ and $\pi_{\color{red}\blacksquare}$ are defined similarly. We shall write $\mathbb{K}^+$ for the smallest group of permutations of $\mathcal{F}$ containing $\pi_{\color{green}\blacksquare}$, $\pi_{\blacksquare}$, $\pi_{\color{red}\blacksquare}$ and all the elements of $\mathbb{K}$.[24]

It turns out that a meridian can be characterized by functions which are "balanced" in a certain sense — and in such a way that any balanced function remains balanced when $\mathcal{F}$ is permuted by any of the functions in $\mathbb{K}^+$. It is the purpose of the present section to show how this is done.

(5.2) Definitions and Notation Two distinct faces will be said to be **opposing** if they are of the same color. Thus ✠ and ★ are opposing faces, ♠ and ♣ are opposing faces, and ♥ and ♦ are opposing faces. For any face F, we shall denote its opposing face by

$$\widehat{\mathsf{F}}\,. \tag{1}$$

Two faces will be said to be **adjacent** if they are distinct and not opposing.

By a **half cube** we shall mean a subset of F consisting of three mutually adjacent faces (thus, one face of each of the three colors green, black and red).

Let M be a set with cardinality at least 4. By a **pre-load** we shall mean a function from any subset of $\mathcal{F}$ into M. A pre-load is a **load** if its domain is all of $\mathcal{F}$. We shall write

$$\mathrm{M}^{\mathcal{F}}$$

for the family of all loads. A pre-load which is constant on a half cube will be said to be singular. If a pre-load is not singular, we shall say that it is **regular**.

For $\{a,b,c,d,e,f\}\subset \mathrm{M}$, we denote the following pre-loads as follows:

$$\ll a,b,c,d \gg \;\equiv\; \{[♥,a],[♦,b],[♠,c],[♣,d]\}\,, \tag{2}$$

$$\ll a,b,c,d,e \gg \;\equiv\; \{[♥,a],[♦,b],[♠,c],[♣,d],[✠,e]\} \tag{3}$$

and

$$\ll a,b,c,d,e,f \gg \;\equiv\; \{[♥,a],[♦,b],[♠,c],[♣,d],[✠,e],[★,f]\}\,. \tag{4}$$

(5.3) Definition and Notation A sub-family $\mathcal{A}$ of $\mathrm{M}^{\mathcal{F}}$ will be said to be a **pre-meridian family of loads for** M if the following conditions are met:[25]

---

[24] The generating set for $\mathbb{K}^+$ may be made much smaller of course.

[25] In view of the second condition, the effect of the first condition is that, if two adjacent sides carry the same value of a given load, that load is balanced if and only if the load is singular. In particular, each singular

$$(\forall\,\{a,b,c,d,e,f\}\subset M\colon\ \#(\{a,b\}\cap\{c,d\})=1)\quad \ll a,b,c,d,e,f\gg\,\in Я \Longleftrightarrow \{a,b\}\cap\{c,d\}\cap\{e,f\}\neq\emptyset\,, \tag{1}$$

$$(\forall\, x\in Я)(\forall\, f\in K^{+})\quad x\in Я \Longleftrightarrow x\circ\phi\in Я \tag{2}$$

and

$$(\forall\,\{a,b,c,d,e\}\subset M\colon\ \ll a,b,c,d,e\gg \text{ is regular})(\exists!\, f\in M)\quad \ll a,b,c,d,e,f\gg\,\in Я\,. \tag{3}$$

For $\ll a,b,c,d,e\gg$ such that there exists a unique element of M such that $\ll a,b,c,d,e,x\gg\,\in Я$, we denote this element x as

$$\boxed{\begin{smallmatrix} & c & \\ a & e & b \\ & d & \end{smallmatrix}}\,. \tag{4}$$

Thus

$$\ll a,b,c,d,e,\boxed{\begin{smallmatrix} & c & \\ a & e & b \\ & d & \end{smallmatrix}}\gg\,\in Я\,. \tag{5}$$

One can visualize the contents of (4) as being attached to all the sides of the cube except the rear one: e is attached to the front ✠, a is attached to the left side ♥, b is attached to the right side ♦, c is attached to the top ♠ and d is attached to the bottom ♣. The operator implicitly defined in (5) will be called a **meridian quinary operator**.

In view of (2), we have

$$\boxed{\begin{smallmatrix} & c & \\ a & e & b \\ & d & \end{smallmatrix}} = \boxed{\begin{smallmatrix} & c & \\ b & e & a \\ & d & \end{smallmatrix}} = \boxed{\begin{smallmatrix} & d & \\ a & e & b \\ & c & \end{smallmatrix}} = \boxed{\begin{smallmatrix} & a & \\ c & e & d \\ & b & \end{smallmatrix}} = \boxed{\begin{smallmatrix} & a & \\ d & e & c \\ & b & \end{smallmatrix}} = \boxed{\begin{smallmatrix} & b & \\ c & e & d \\ & a & \end{smallmatrix}}\,. \tag{6}$$

Suppose $\{a,b,c,d,x\}\subset M$ and $\{a,b\}\neq\{c,d\}$. If $\{a,b\}\cap\{c,d\}=\emptyset$, then $\ll a,b,c,d,x\gg$ is regular and so (3) implies that there exists a unique $y\in M$ such that $\ll a,b,c,d,x,y\gg\,\in Я$. If $u\in M$, $\{a,b\}\cap\{c,d\}=\{u\}$ and $x\neq u$, then (1) implies that u is the unique element of M such that $\ll a,b,c,d,x,u\gg\,\in Я$. It follows that

$$(\forall\,\{a,b,c,d\}\subset M\colon\ \{a,b\}\neq\{c,d\})\quad (M\triangle(\{a,b\}\cap\{c,d\}))\ni x\hookrightarrow \boxed{a,b;c,d}\,(x)\equiv\boxed{\begin{smallmatrix} & c & \\ a & x & b \\ & d & \end{smallmatrix}}\in M \tag{7}$$

is well-defined.[26]

For $\{a,b\}\subset M$ we denote

$$M_{(a,b)}\ \equiv\ M\triangle\{a,b\}\,. \tag{8}$$

A pre-meridian family Я of loads for M will be said to be a **meridian family of loads for** M provided that

$$\begin{gathered}(\forall\,\{a,b\}\subset M)\quad M_{(a,b)}\times M_{(a,b)}\times M_{(a,b)}\ni\lceil r,s,t\rceil\hookrightarrow \lfloor r,t,s\rfloor^{a}_{b}\equiv\boxed{\begin{smallmatrix} & r & \\ a & t & b \\ & s & \end{smallmatrix}}\in M_{(a,b)} \\ \lfloor,,\rfloor^{a}_{b} \text{ is a libra operator on } M_{(a,b)}\end{gathered} \tag{9}$$

and

$$(\forall\,\{a,b,c,d\}\subset M\colon\ \{a,b\}\cap\{c,d\}=\emptyset)(\forall\, x\,|\,\mathcal{F}\hookrightarrow M)\quad x\in Я \Longleftrightarrow \boxed{a,b;c,d}\circ x\in Я\,. \tag{10}$$

---

load is balanced.

[26] The terminology $\boxed{a,b;c,d}$ is identical with that of (2.3). This constitutes a venial solecism however, since we shall see *infra* that they mean the same when they occur in a common context.

We note that for a meridian family of loads, it follows from (6) that the libra $M_{(a,b)}$ specified in (9) is abelian.

(5.4) Theorem Let Я be a meridian family of loads for M. We define

$$\mathcal{M} \equiv \{\boxed{a,b;c,d} : \{a,b,c,d\} \subset M \quad \text{and} \quad \{a,b\} \cap \{c,d\} = \emptyset\}. \tag{1}$$

Then $\mathcal{M}$ is a meridian family of involutions of M.

Proof. We first show that the elements of $\mathcal{M}$ are involutions. Let $\boxed{a,b;c,d}$ be in $\mathcal{M}$ for $\{a,b,c,d\} \subset M$. From (5.3.9), we have

$$(\forall\, x \in M_{(a,b)}) \quad \boxed{a,b;c,d} \circ \boxed{a,b;c,d}(x) = \lfloor c, \lfloor c,x,d \rfloor^a_b, d \rfloor^a_b$$

Since $M_{(a,b)}$ is abelian, we have $\lfloor c,x,d \rfloor^a_b = \lfloor d,x,c \rfloor^a_b$ and so the above becomes

$$(\forall\, x \in M_{(a,b)}) \quad \boxed{a,b;c,d} \circ \boxed{a,b;c,d}(x) \overset{\text{by (4.12.1)}}{=\!=\!=} \lfloor c, \lfloor d,x,c \rfloor^a_b, d \rfloor^a_b \overset{\text{by (4.3.1)}}{=\!=\!=} \lfloor \lfloor c,c,x \rfloor^a_b, d, d \rfloor^a_b \overset{\text{by (4.2.1)}}{=\!=\!=} = x. \tag{2}$$

In view of (5.3.2) we have

$$\boxed{a,b;c,d} = \boxed{c,d;a,b} \tag{3}$$

and so, as above,

$$(\forall\, x \in M_{(c,d)}) \quad \boxed{a,b;c,d} \circ \boxed{a,b;c,d}(x) = x$$

which, with (3), implies that $\boxed{a,b;c,d}$ is an involution.

Next we prove (2.3.2). Let $\{a,b,d,e\} \subset M$ be such that $\{a,e\} \cap \{b,d\} = \emptyset$. We have

$$\boxed{a,e;b,d}(b) = \lfloor b,b,d \rfloor^a_e = d \quad \text{and} \quad \boxed{a,e;b,d}(d) = \lfloor b,d,d \rfloor^a_e = b. \tag{4}$$

From (3) and (4) follows that

$$\boxed{a,e;b,d}(a) = e \quad \text{and} \quad \boxed{a,e;b,d}(e) = a. \tag{5}$$

Now suppose that $\phi \in \mathcal{M}$ also satisfies

$$\phi(a) = e, \quad \phi(e) = a, \quad \phi(b) = d \quad \text{and} \quad \phi(d) = b. \tag{6}$$

Choose $\{r,s,u,v\} \subset M$ such that $\{r,s\} \cap \{u,v\} = \emptyset$ and $\phi = \boxed{r,v;s,u}$. We need to show that $\boxed{r,v;s,u} = \boxed{a,e;b,d}$ so, without loss of generality, we may suppose that $s \notin \{a,e,b,d\}$. As we are dealing with an involution, since $\boxed{r,v;s,u}(s) = u$, it follows that $u \notin \{a,e,b,d\}$ as well. We have

$$\lfloor a,a,e \rfloor^s_u = e \quad \text{and} \quad \lfloor r,a,v \rfloor^s_u = \boxed{r,v;s,u}(a) = \phi(a) = e.$$

By (4.13.2) it follows that $\lfloor a,x,e \rfloor^s_u = \lfloor r,x,v \rfloor^s_u$ for all $x \in M$. This just means that $\phi = \boxed{r,v;s,u} = \boxed{a,e;s,u}$. In similar manner we show that $\boxed{a,e;b,d} = \boxed{a,e;s,u}$. Putting these both together we obtain that

$$\phi = \boxed{r,v;s,u} = \boxed{a,e;b,d}$$

which establishes (2.3.2).

Now we turn to (2.3.3). Let $\{a,d\} \subset M$ and let α, β and γ be elements of $\mathcal{M}$, each of which sends a to d. Let $b \in M$ be distinct from a and define

$$e \equiv \alpha(b)\,, \quad m \equiv \beta(b) \quad \text{and} \quad n \equiv \gamma(b)\,.$$

By (2.3.2) we know that

$$\alpha = \boxed{a,d;b,e}\,, \quad \beta = \boxed{a,d;b,m} \quad \text{and} \quad \gamma = \boxed{a,d;b,n}\,.$$

For $x \in M$ we have

$$\beta\circ\gamma(x) = \boxed{\begin{array}{c} b \\ a \ \boxed{\begin{array}{c} b \\ a\ x\ d \\ n \end{array}} \ d \\ m \end{array}} = \lfloor b, \lfloor b,x,n \rfloor_d^a ,m \rfloor_d^a = \lfloor \lfloor b,b\ ,x \rfloor_d^a ,n,m \rfloor_d^a = \lfloor x,n,m \rfloor_d^a$$

whence follows that

$$\alpha\circ\beta\circ\gamma(x) = \boxed{a,d;b,e}(\lfloor x,n,m \rfloor_d^a) = \lfloor b, \lfloor x,n,m \rfloor_d^a ,e \rfloor_d^a \overset{\text{by (4.3.1) and by (4.2.1)}}{=\!=\!=\!=\!=\!=\!=} \lfloor b,x, \lfloor n,m,e \rfloor_d^a \rfloor_d^a .$$

If $c \equiv \lfloor n,m,e \rfloor_d^a$, this just means that $\alpha\circ\beta\circ\gamma = \boxed{a,d;b,c}$, whence follows the conclusion of (2.3.3).

Finally we turn to (2.3.1). Let α and β be elements of $\mathcal{M}$. Then there exists $\{a,b,d,e,r,s,u,v\} \subset M$ such that

$$\alpha = \boxed{a,e;b,d} \quad \text{and} \quad \beta = \boxed{r,v;s,u}\,.$$

For any $x \in M$ we have

$$\boxed{a,e;b,d}\circ\boxed{r,v;s,u}(x) = \boxed{\begin{array}{c} a \\ b \ \boxed{\begin{array}{c} r \\ s\ x\ u \\ v \end{array}} \ d \\ e \end{array}} \overset{\text{by (5.3.10)}}{=\!=\!=\!=}$$

$$\boxed{\begin{array}{ccc} & \boxed{\begin{array}{c} a \\ b\ r\ d \\ e \end{array}} & \\ \boxed{\begin{array}{c} a \\ b\ s\ d \\ e \end{array}} & \boxed{\begin{array}{c} a \\ b\ x\ d \\ e \end{array}} & \boxed{\begin{array}{c} a \\ b\ u\ d \\ e \end{array}} \\ & \boxed{\begin{array}{c} a \\ b\ v\ d \\ e \end{array}} & \end{array}}\,.$$

Setting

$$d \equiv \boxed{\begin{array}{c} a \\ b\ r\ d \\ e \end{array}}\,, \quad n \equiv \boxed{\begin{array}{c} a \\ b\ s\ d \\ e \end{array}}\,, \quad p \equiv \boxed{\begin{array}{c} a \\ b\ u\ d \\ e \end{array}}\,, \quad q \equiv \boxed{\begin{array}{c} a \\ b\ v\ d \\ e \end{array}} \quad \text{and} \quad y \equiv \alpha(x)$$

we obtain

$$\alpha\circ\beta(x) = \boxed{\begin{array}{c} m \\ n\ y\ p \\ q \end{array}} = \boxed{m,q;n,p}\circ\alpha(x)\,.$$

It follows that $\alpha\circ\beta\circ\alpha = \boxed{m,q;n,p}$, which establishes (2.3.1). Q.E.D.

(5.5) Example Let L be the example of (2.7): a line meridian. Then a load $x \,|\, \mathcal{F} \hookrightarrow M$ is in Я if, and only if {{x(♥),x(♦)},{x(♠),x(♣)},{x(✠),x(★)}} is a cubic triple of pairs of points of L.

6. Wurfs and the Cross Ratio

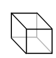
(6.1) Introduction The **cross ratio** of four elements w, x, y and z of a field F is defined to be

$$\frac{(w-y)(x-z)}{(x-y)(w-z)}, \tag{1}$$

which makes sense whenever no more than two of the elements are identical, and even when the value ∞ is permitted. Its use dates back to antiquity, but it seems first in 1847 (by Karl von Staudt) to be seriously considered in the context of a projective line, or what is here called a meridian. Since the cross ratio is invariant under linear fractional transformations, its avatar in a meridian must be invariant under homographies. Von Staudt's idea was to consider ordered quadruples of points on a line, regarding any two such quadruples as "equal", provided there exists a homography which maps the coordinates of one to the corresponding coordinates of another. His term for such a "quadruple" was a "Wurf". This German word means a "throw" or "cast" in English. We shall retain the term "Wurf" in this paper, in part because of its closer relation to a cube, which we shall find to be of some use in the exposition here – but since this paper is written in English, we shall drop the upper case "W" in favour of "w".

In the present section we shall show how the meridian may be viewed in the context of wurfs and the cross ratio.

(6.2) The Cube The word "Wurf" in German evokes its derivative "Würfel", which literally means a die[27] or derivatively, a cube. In fact, a cube



offers a suggestive setting to manifest von Staudt's idea. We proceed to analyze a cube in detail.

It has 8 **vertices**, 12 **edges** and 6 **faces**. Each vertex **p** is connected to three other **adjacent vertices** directly via edges, and to a fourth vertex, its **opposite vertex** **o**, which is unique in the sense that **p** and **o** are on no common face. We shall call this set consisting of the three vertices adjacent to **p** and the vertex opposite to **p** the **cubic quadriad exclusive of p**. The complement of this set will be called the **cubic quadriad of p**. This latter terminology is justified because, for any element **q** of the cubic quadriad exclusive of **p**, the cubic quadriad exclusive of **q** is precisely the cubic quadriad including **p**. Thus the set of vertices of a cube is a disjoint union of the sets of vertices of two cubic quadriads.

We shall label the elements of one cubic quadriad **a**, **b**, **c** and **d**. The above considerations suggest that we then label the other cubic quadriad as **a**, **b**, **c** and **d** where **a** is the opposite vertex of **a**, **b** the opposite vertex of **b** and so on. We denote

$$\mathbf{Q} \equiv \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}\,, \quad \mathbf{Q} \equiv \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\} \quad \text{and} \quad \mathbf{K} \equiv \mathbf{Q}\cup\mathbf{Q}\,. \tag{1}$$

Any two elements of a cubic quadriad determine a face: we shall write ✠ for the face determined such that

$$\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in ✠\,. \tag{2}$$

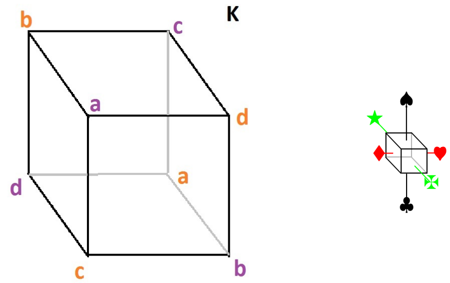
Then the face ★ opposite to ✠ satisfies

$$\mathbf{c},\mathbf{d},\mathbf{a},\mathbf{b} \in ★\,. \tag{3}$$

We define the notation of the other four faces as follows:

[27] The singular form of the noun "dice".

$$\mathbf{a},\mathbf{c},\mathbf{b},\mathbf{d} \in \spadesuit, \quad \mathbf{b},\mathbf{d},\mathbf{a},\mathbf{c} \in \clubsuit, \quad \mathbf{a},\mathbf{d},\mathbf{b},\mathbf{c} \in \blacklozenge, \quad \mathbf{b},\mathbf{c},\mathbf{a},\mathbf{d} \in \heartsuit \,. \tag{4}$$



We now examine the permutations of the vertices of **K** which leave the cube intact. They can be separated into two general types: those which leave the cubic quadriads invariant and those which interchange the cubic quadriads. These permutations may all be realized by rotations around certain axes. By a **vertex axis** we mean the a line segment with opposite vertices as endpoints. By an **edge axis** we mean a line segment through the midpoints of opposite edges. By a **face axis** we mean a line segment with endpoints the centers of opposite faces.

By a **face involution** we shall mean a rotation of 180^o about a face axis. The involutions corresponding to the face axes through ✠ and ★, through ♠ and ♣ and that through ♥ and ♦, respectively, will be denoted

$$\sigma_{\blacksquare}\,, \quad \sigma_{\blacksquare} \quad \text{and} \quad \sigma_{\blacksquare}\,, \tag{5}$$

respectively. These three involutions of the cube leave the cubic quadriads invariant and, along with the identity permutation $\iota_{\mathbf{K}}$ of the cube, form a group $\mathfrak{K}_4$ called the **Klein 4-group**. A 90^o rotation around a face axis will be called a **face rotation**. The face rotation which leaves ✠ and ★ invariant and which carries **a** to **d** will be written

$$\rho_{(\mathbf{ad})} \tag{6}$$

The other six such rotations are defined analogously:

$$\rho_{(\mathbf{ac})}, \quad \rho_{(\mathbf{ac})}, \quad \rho_{(\mathbf{ab})}, \quad \rho_{(\mathbf{ab})} \quad \text{and} \quad \rho_{(\mathbf{ad})}\,. \tag{7}$$

These face rotations interchange cubic quadriads.

Rotating the cube 180^o about an edge axis performs what we shall call an **edge involution**. If we take the edge axis which pierces the midpoint of the edge joining **a** to **b** and the midpoint of the opposite edge (which joins **b** to **a**), shall write the corresponding edge involution as

$$\epsilon_{(\mathbf{ab})}\,. \tag{8}$$

The other edge involutions are denoted analogously

$$\epsilon_{(\mathbf{ac})}, \quad \epsilon_{(\mathbf{ad})}, \quad \epsilon_{(\mathbf{bc})}, \quad \epsilon_{(\mathbf{bd})} \quad \text{and} \quad \epsilon_{(\mathbf{cd})}\,. \tag{9}$$

We call the vertex axis through **a** and **a**

$$\overline{\mathbf{a},\mathbf{a}}\,. \tag{10}$$

The other vertex axes are denoted analogously

$$\overline{\mathbf{b},\mathbf{b}}, \quad \overline{\mathbf{c},\mathbf{c}} \quad \text{and} \quad \overline{\mathbf{d},\mathbf{d}}\,. \tag{11}$$

The 120^o rotations about the vertex axes leave the cubic quadriads fixed. The one about the vertex axis $\overline{\mathbf{a},\mathbf{a}}$ which sends **b** to **c** (and **b** to **c**) will be denoted

$$\theta_{(\mathbf{a;bc})}\,. \tag{12}$$

The other such rotations are denoted analogously

$$\theta_{(\mathbf{a;cb})},\quad \theta_{(\mathbf{b;cd})},\quad \theta_{(\mathbf{b;dc})},\quad \theta_{(\mathbf{c;da})},\quad \theta_{(\mathbf{c;ad})},\quad \theta_{(\mathbf{d;ab})}\quad \text{and}\quad \theta_{(\mathbf{d;ba})}\,. \tag{13}$$

In all, there are twenty-four of these transformations of the cube. The group $\mathfrak{K}_4$ is normal in this larger group of rotational transformations. This means that if we compose each element of $\mathfrak{K}_4$ with a fixed transformation ϕ, the resulting set of four elements will be the same whether the composition takes ϕ before or after. Such a set is called a coset of $\mathfrak{K}_4$, and we shall list $\mathfrak{K}_4$ and the five other cosets of $\mathfrak{K}_4$ as a means of describing the transformations of the cube:

$$\{\iota,\sigma_{\textcolor{red}{\blacksquare}},\sigma_{\textcolor{green}{\blacksquare}},\sigma_{\blacksquare}\}\,, \tag{14}$$

$$\{\theta_{(\mathbf{a;bc})},\theta_{(\mathbf{b;cd})},\theta_{(\mathbf{c;da})},\theta_{(\mathbf{d;ba})}\}\,, \tag{15}$$

$$\{\theta_{(\mathbf{a;cb})},\theta_{(\mathbf{b;dc})},\theta_{(\mathbf{c;ad})},\theta_{(\mathbf{d;ab})}\}\,, \tag{16}$$

$$\{\rho_{(\mathbf{ab})},\rho_{(\mathbf{ac})},\epsilon_{(\mathbf{bc})},\epsilon_{(\mathbf{ad})}\}\,, \tag{17}$$

$$\{\rho_{(\mathbf{ac})},\rho_{(\mathbf{ad})},\epsilon_{(\mathbf{ab})},\epsilon_{(\mathbf{cd})}\}\,, \tag{18}$$

$$\{\rho_{(\mathbf{ad})},\rho_{(\mathbf{ab})},\epsilon_{(\mathbf{ac})},\epsilon_{(\mathbf{bd})}\}\,. \tag{19}$$

The permutations in each of the sets (14), (15) and (16) send each cubic quadric to itself, while permutations in the latter three sets interchange them. Thus the union of the first three is a normal subgroup. We shall denote the group of these 24 permutations of the cube by

$$\mathfrak{K}\,. \tag{20}$$

Let F denote the set of face axes, E the set of edge axes and V the set of vertex axes. Each of these sets is left invariant by the cube transformations described above, which fact induces groups of permutations on these sets. Since F has cardinality 3 and thus is only subject to six permutations, the correspondence from the cube transformations is not injective. The set E has six elements and so is subject to 6! permutations: here the correspondence is not surjective. However V has cardinality 4 and the correspondence is a bijection of the group of physical transformations of the cube onto the group of permutations of V. For the remainder of this section we shall commit a minor abuse of notation by viewing the elements of $\mathfrak{K}$ as permutations of the set V rather than the set $\mathbf{K}$ of vertices of the cube.

(6.3) Definition Let M be any set of cardinality at least 4. By an M-**quadriad** we shall mean a function $x\big|\mathsf{V}\hookrightarrow \mathrm{M}$ such that the cardinality of the range is at least 3. Now let $\mathcal{G}$ be any 3-transitive group of permutations of M. We define a an equivalence relation $\sim$ on the set $\mathcal{W}$ of all M-quadriads by

$$(\forall\,\{x,y\}\subset\mathcal{W})\quad x\sim y\quad \Longleftrightarrow\quad (\exists\,\sigma\in\mathfrak{K}_4 \text{ and } \phi\in\mathcal{G})\quad y=\phi\circ x\circ\sigma\,. \tag{1}$$

We write the family of all equivalence classes relative to $\sim$ as $\mathfrak{W}$. An element of $\mathfrak{W}$ will be said to be a **wurf**. If one member of a wurf has a range of cardinality 3, then all members of that wurf have ranges of cardinality 3. Such wurfs will be called **singular** while all other wurfs will be called **regular**. There are exactly three distinct singular wurfs:

$$\widetilde{\textcolor{green}{\blacksquare}} \equiv \{x\in\mathcal{W}\colon \text{either}\quad x(\overline{\mathbf{a},\mathbf{a}})=\overline{\mathbf{b},\mathbf{b}}\quad \text{or}\quad x(\overline{\mathbf{c},\mathbf{c}})=\overline{\mathbf{d},\mathbf{d}}\}\,, \tag{2}$$

$$\widetilde{\blacksquare} \equiv \{x\in\mathcal{W}\colon \text{either}\quad x(\overline{\mathbf{a},\mathbf{a}})=\overline{\mathbf{c},\mathbf{c}}\quad \text{or}\quad x(\overline{\mathbf{b},\mathbf{b}})=\overline{\mathbf{d},\mathbf{d}}\} \tag{3}$$

and $$\tilde{\color{red}\blacksquare} \equiv \{x\in\mathcal{W}\colon \text{either} \quad x(\overline{\mathbf{a},\mathbf{a}}) = \overline{\mathbf{d},\mathbf{d}} \quad \text{or} \quad x(\overline{\mathbf{b},\mathbf{b}}) = \overline{\mathbf{c},\mathbf{c}}\}\,. \tag{4}$$

For $\{a,b,c,d\}\subset M$ and $\{\mathbf{б},\mathbf{д},\mathbf{л},\mathbf{ч}\} = V$, we shall at times use the notation

$$\left\langle\begin{smallmatrix} \mathbf{б} & \mathbf{д} & \mathbf{л} & \mathbf{ч} \\ a & b & c & d \end{smallmatrix}\right\rangle \equiv \{[\mathbf{б},a],[\mathbf{д},b],[\mathbf{л},c],[\mathbf{ч},d]\}\,. \tag{5}$$

and the abbreviation

$$<a,b,c,d> \equiv \{[\overline{\mathbf{a},\mathbf{a}}\,,a],[\overline{\mathbf{b},\mathbf{b}}\,,b],[\overline{\mathbf{c},\mathbf{c}}\,,c],[\overline{\mathbf{d},\mathbf{d}}\,,d]\}\,. \tag{6}$$

When employing the above notation for a wurf, we write

$$<a,b,c,d>^{\sim} \tag{7}$$

for the element of $\mathfrak{W}$ of which $<a,b,c,d>$ is a member. In terms of this notation we have

$$\tilde{\color{green}\blacksquare} = \{<a,b,c,d>: a=b \quad \text{or} \quad c=d\}\,, \tag{8}$$

$$\tilde{\blacksquare} = \{<a,b,c,d>: a=c \quad \text{or} \quad b=d\} \tag{9}$$

and $$\tilde{\color{red}\blacksquare} = \{<a,b,c,d>: a=d \quad \text{or} \quad b=c\,.\} \tag{10}$$

(6.4) Theorem Let $\{a,b,c\}\subset M$ be a basis for M (no two elements of $\{a,b,c\}$ are equal). Then

$$(\forall\, \mathfrak{x}\in\mathfrak{W})(\exists\, t\in M) \quad <a,b,c,t>\in\mathfrak{x}\,. \tag{1}$$

Furthermore,

$$\text{if } \mathfrak{x}\in\{\tilde{\color{green}\blacksquare},\tilde{\blacksquare},\tilde{\color{red}\blacksquare}\}, \text{ then the t in (1) is unique.} \tag{2}$$

Proof. [Case: $\mathfrak{x}\in\tilde{\color{green}\blacksquare}$] Let $t\equiv c$. If follows from (6.3.8) that (1) holds and b is the only value for t for which it does hold.

[Case: $\mathfrak{x}\in\tilde{\blacksquare}$] Let $t\equiv b$. If follows from (6.3.9) that (1) holds and b is the only value for t for which it does hold.

[Case: $\mathfrak{x}\in\tilde{\color{red}\blacksquare}$] Let $t\equiv a$. If follows from (6.3.10) that (1) holds and b is the only value for t for which it does hold.

[Case: $\mathfrak{x}\notin\{\tilde{\color{green}\blacksquare},\tilde{\blacksquare},\tilde{\color{red}\blacksquare}\}$] Let $<p,q,r,s>$ be any wurf in $\mathfrak{x}$. Since $<p,q,r,s>$ is regular, $\boxed{\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}}$ is an element of $\mathcal{G}$. We have

$$\boxed{\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}}\circ<p,q,r,s>\in\mathfrak{x} \quad \Longrightarrow <a,b,c,\boxed{\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}}(s)>\in\mathfrak{x}\,. \tag{3}$$

Thus we may let $t\equiv\boxed{\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}}(s)$. Q.E.D.

(6.5) Notation Let $\mathcal{G}$ be a 3-transitive group of permutations on a set M, where M has at least four elements. It follows from Theorem (6.4) that, for any basis $\{x,y,z\}$ for M, the function

$$M\ni t\hookrightarrow <x,y,z,t>^{\sim}\in\mathfrak{W} \text{ is surjective.} \tag{1}$$

We shall shall denote the function of (1) by

$$\widetilde{\boxed{\tfrac{x}{a,a}\ \tfrac{y}{b,b}\ \tfrac{z}{c,c}}}\ . \tag{2}$$

(6.6) Theorem Let $\mathcal{G}$ be a 3-transitive group of permutations on a set M, where M has at least four elements. Then the following statements are equivalent:

$$(\forall\,\{a,b,c\}\text{ a basis in M})\quad \widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}\Big|\mathrm{M}\hookrightarrow\mathfrak{W}\quad\text{is a bijection,} \tag{1}$$

and

$$(\forall\,\phi\in\mathcal{G}\colon\ (\exists\,\{a,b\}\subset \mathrm{M})\ a\neq b,\ \phi(a)=b\text{ and }\phi(b)=a)\quad \phi\in\mathcal{G}_2\,. \tag{2}$$

Proof. [(1) $\Longleftarrow$ (2)] Suppose first that (2) is satisfied, and let $\{s,t\}\subset \mathrm{M}$ be such that

$$\widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}(s)\ =\ \widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}(t)\ . \tag{3}$$

If $\widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}(t)\in\{\widetilde{{\color{green}\blacksquare}},\widetilde{\blacksquare},\widetilde{{\color{red}\blacksquare}}\}$, it follows from (6.4.2) that $s=t$. If $\widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}(t)\notin\{\widetilde{{\color{green}\blacksquare}},\widetilde{\blacksquare},\widetilde{{\color{red}\blacksquare}}\}$, then $\{s,t\}\cap\{a,b,c\}=\emptyset$, and so we shall presume that this is so. There exist $\phi\in\mathcal{G}$ and $\sigma\in\mathfrak{K}_4$ such that

$$\langle a,b,c,s\rangle=\phi\circ\langle a,b,c,t\rangle\circ\sigma\,. \tag{4}$$

If $\sigma=\iota_V$, then ϕ and ι_M agree on three distinct points, whence follows that they are identical, and so

$$s=\phi(t)=t\,. \tag{5}$$

If $\sigma=\sigma_{\color{green}\blacksquare}$, then (4) implies

$$\langle a,b,c,s\rangle=\phi\circ\langle b,a,t,c\rangle=\langle\phi(b),\phi(a),\phi(t),\phi(c)\rangle\,. \tag{6}$$

It follows in particular that $a=\phi(b)$ and $b=\phi(a)$, which by (2) implies that $\phi=\phi^{-1}$. Consequently from (6) we have

$$c=\phi(t)\quad\Longrightarrow\quad t=\phi(c)\quad\Longrightarrow\quad t=s\,. \tag{7}$$

If $\sigma=\sigma_{\blacksquare}$ or $\sigma=\sigma_{\color{red}\blacksquare}$, arguments analogous to that just used show that $s=t$ in each case. Hence, in view of (7), it follows that then function $\widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}$ is injective, and so bijective.

[(1) $\Longrightarrow$ (2)] We now suppose that (1) holds. Let a and b be distinct elements of M and ϕ an element of $\mathcal{G}$ such that

$$\phi(a)=b\quad\text{and}\quad\phi(b)=a\,. \tag{8}$$

Let t be any element of M not in $\{a,b\}$. Let $c\equiv\phi(t)$ and assume that $t\neq\phi(c)$. Then

$$\widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}(t)=\langle a,b,c,t\rangle^{\sim}=(\phi\circ\langle a,b,c,t\rangle)^{\sim}=\langle b,a,\phi(c),c\rangle^{\sim}=$$

$$(\langle b,a,\phi(c),c\rangle\circ\sigma_{\color{green}\blacksquare})^{\sim}=\langle a,b,c,\phi(c)\rangle^{\sim}=\widetilde{\boxed{\tfrac{a}{a,a}\ \tfrac{b}{b,b}\ \tfrac{c}{c,c}}}(\phi(c))$$

which violates (1). It follows that $t=\phi(c)$, which implies that $\phi\in\mathcal{G}_2$. Q.E.D.

(6.7) Corollary Let M be a set with at least four elements and let $\mathcal{G}$ be a 3-transitive group of permutations of M such that (6.6.2) holds. Define

$$\mathfrak{G} \equiv \left\{ \widetilde{\boxed{\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathbf{a,a} & \mathbf{b,b} & \mathbf{c,c} \end{smallmatrix}}} \circ \widetilde{\boxed{\begin{smallmatrix} \mathrm{u} & \mathrm{v} & \mathrm{w} \\ \mathbf{a,a} & \mathbf{b,b} & \mathbf{c,c} \end{smallmatrix}}}^{-1} : \{\mathrm{a,b,c}\} \text{ and } \{\mathrm{u,v,w}\} \text{ are bases for M} \right\} \tag{1}$$

Then $\mathfrak{G}$ is a group isomorphic with $\mathcal{G}$ and M is isomorphic with $\mathfrak{W}$ in the sense that there exists a bijection $\psi \big| \mathrm{M} \hookrightarrow \mathfrak{W}$ such that

$$\mathcal{G} \;=\; \{\psi^{-1} \circ \mathfrak{f} \circ \psi : \mathfrak{f} \in \mathfrak{G}\}\,. \tag{2}$$

Proof. Let a, b and c be pairwise distinct elements of M and define ψ to be $\widetilde{\boxed{\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathbf{a,a} & \mathbf{b,b} & \mathbf{c,c} \end{smallmatrix}}}$. Q.E.D.

(6.8) Discussion One readily checks that, for any ϕ in $\mathfrak{K}$,

$$(\forall\, \mathrm{x} \text{ and } \mathrm{y} \text{ M-quadriads}) \quad \phi(\mathrm{x}) \sim \phi(\mathrm{y}) \Longleftrightarrow \mathrm{x} \sim \mathrm{y}\,. \tag{1}$$

Thus, to each $\phi \in \mathfrak{K}$ corresponds a unique function

$$\widetilde{\phi} \big| \mathfrak{W} \hookrightarrow \mathfrak{W} \quad \text{such that } (\forall\, \mathfrak{w} \in \mathfrak{W})(\forall\, \mathrm{x} \in \mathfrak{w}) \quad \mathrm{x} \circ \phi \in \widetilde{\phi}(\mathrm{x})\,. \tag{2}$$

It follows from the definition of the equivalence relation $\sim$ that

$$(\forall\, \sigma \in \mathfrak{K}_4) \quad \widetilde{\sigma} = \iota_{\mathfrak{W}}\,. \tag{3}$$

Consequently, if α and β are elements of the same coset (where the cosets are given in (6.2.14)-(6.2.19), then

$$\widetilde{\alpha} = \widetilde{\beta}\,. \tag{4}$$

Thus, if we choose one element from each coset, we shall obtain a group of permutations $\mathfrak{W}$:

$$\{\iota, \widetilde{\theta}_{(\mathbf{a;bc})}, \widetilde{\theta}_{(\mathbf{a;cb})}, \widetilde{\rho}_{(\mathbf{ab})}, \widetilde{\rho}_{(\mathbf{ac})}, \widetilde{\rho}_{(\mathbf{ad})}\}\,. \tag{5}$$

For reference we compute for any M-quadriad $<\mathrm{a,b,c,d}>$

$$<\mathrm{a,b,c,d}> \circ\, \theta_{(\mathbf{a;bc})} = <\mathrm{a,c,d,b}> \quad \text{and} \quad \widetilde{\theta}_{(\mathbf{a;bc})}(<\mathrm{a,b,c,d}>^{\sim}) = <\mathrm{a,c,d,b}>^{\sim}\,, \tag{6}$$

$$<\mathrm{a,b,c,d}> \circ\, \theta_{(\mathbf{a;cb})} = <\mathrm{a,d,b,c}> \quad \text{and} \quad \widetilde{\theta}_{(\mathbf{a;cb})}(<\mathrm{a,b,c,d}>^{\sim}) = <\mathrm{a,d,b,c}>^{\sim}\,, \tag{7}$$

$$<\mathrm{a,b,c,d}> \circ\, \rho_{(\mathbf{ab})} = <\mathrm{b,d,a,c}> \quad \text{and} \quad \widetilde{\rho}_{(\mathbf{ab})}(<\mathrm{a,b,c,d}>^{\sim}) = <\mathrm{b,d,a,c}>^{\sim}\,, \tag{8}$$

$$<\mathrm{a,b,c,d}> \circ\, \rho_{(\mathbf{ac})} = <\mathrm{c,d,b,a}> \quad \text{and} \quad \widetilde{\rho}_{(\mathbf{ab})}(<\mathrm{a,b,c,d}>^{\sim}) = <\mathrm{c,d,b,a}>^{\sim}\,, \tag{9}$$

and

$$<\mathrm{a,b,c,d}> \circ\, \rho_{(\mathbf{ad})} = <\mathrm{d,a,b,c}> \quad \text{and} \quad \widetilde{\rho}_{(\mathbf{ad})}(<\mathrm{a,b,c,d}>^{\sim}) = <\mathrm{d,a,b,c}>^{\sim}\,. \tag{10}$$

Also for reference we compute for any M-quadriad $<\mathrm{a,b,c,d}>$

$$<\mathrm{a,b,c,d}> \circ\, \sigma_{\blacksquare} = <\mathrm{d,a,c,b}>\,, \tag{11}$$

$$<\mathrm{a,b,c,d}> \circ\, \sigma_{\blacksquare} = <\mathrm{c,d,a,b}> \tag{12}$$

and

$$<\mathrm{a,b,c,d}> \circ\, \sigma_{\blacksquare} = <\mathrm{b,a,d,c}>\,. \tag{13}$$

(6.9) Theorem Let M be a set with at least four elements and let $\mathcal{G}$ be a 3-transitive group of permutations of M such that (6.6.2) holds. Then

$$(\forall\, \phi \in \mathfrak{W}) \quad \widetilde{\phi} \in \mathfrak{G}\,. \tag{1}$$

Proof. We shall prove the theorem for $\phi = \widetilde{\rho}_{(\mathbf{ad})}$. The proofs for the other cases are analogous.

Let {a,b,c} be a basis for M. Let $\mathfrak{x}$ be any element of $\mathfrak{W}$. By (6.6.1) there exists a unique t such that <a,b,c,t>$\in\mathfrak{x}$. We have

$$\widetilde{\rho}_{(\mathbf{ad})}(\mathfrak{x}) = \widetilde{\rho}_{(\mathbf{ad})}(\langle a,b,c,t\rangle^{\sim}) \xlongequal{\text{by (6.8.10)}} \langle t,a,b,c\rangle^{\sim} \xlongequal{\text{by (6.8.11)}}$$

$$\langle t,a,b,c\rangle^{\sim}\circ\sigma_{\color{red}\blacksquare} = \langle c,b,a,t\rangle^{\sim} = \left(\boxed{\begin{smallmatrix} a & b & c\\ c & b & a\end{smallmatrix}}\circ\langle a,b,c,\boxed{\begin{smallmatrix} a & b & c\\ c & b & a\end{smallmatrix}}(t)\rangle\right)^{\sim} =$$

$$\widetilde{\boxed{\tfrac{c}{a,a}\ \tfrac{b}{b,b}\ \tfrac{a}{c,c}}}\circ\boxed{\begin{smallmatrix} a & b & c\\ c & b & a\end{smallmatrix}}\circ\left(\widetilde{\boxed{\tfrac{c}{a,a}\ \tfrac{b}{b,b}\ \tfrac{a}{c,c}}}\right)^{-1}(\langle a,b,c,t\rangle^{\sim}) = \widetilde{\boxed{\tfrac{c}{a,a}\ \tfrac{b}{b,b}\ \tfrac{a}{c,c}}}\circ\boxed{\begin{smallmatrix} a & b & c\\ c & b & a\end{smallmatrix}}\circ\left(\widetilde{\boxed{\tfrac{c}{a,a}\ \tfrac{b}{b,b}\ \tfrac{a}{c,c}}}\right)^{-1}(\mathfrak{x}).$$

Consequently $\widetilde{\rho}_{(\mathbf{ad})} = \widetilde{\boxed{\tfrac{c}{a,a}\ \tfrac{b}{b,b}\ \tfrac{a}{c,c}}}\circ\boxed{\begin{smallmatrix} a & b & c\\ c & b & a\end{smallmatrix}}\circ\left(\widetilde{\boxed{\tfrac{c}{a,a}\ \tfrac{b}{b,b}\ \tfrac{a}{c,c}}}\right)^{-1}$. It follows from (6.7.2) that the latter is in $\mathfrak{G}$. Q.E.D.

(6.10) Theorem Let M be a set with at least four elements and let $\mathcal{G}$ be a 3-transitive group of permutations of M. Then necessary and sufficient conditions for $\mathcal{G}$ to be a meridian group of permutations are

$$(\forall\,\phi\in\mathcal{G}\colon\ (\exists\,\{a,b\}\subset M)\ a\neq b,\ \phi(a)=b \text{ and } \phi(b)=a)\quad \phi\in\mathcal{G}_2 \tag{1}$$

and

$$(\exists!\,\omega\in\mathcal{G}_4)\quad \omega(\widetilde{\color{red}\blacksquare}) = \widetilde{\color{green}\blacksquare} \quad\text{and}\quad \omega(\widetilde{\color{green}\blacksquare}) = \widetilde{\blacksquare}. \tag{2}$$

Proof. Suppose first that $\mathcal{G}$ is a meridian group of permutations of M. Recalling that M is meridian isomorphic with $\mathfrak{W}$, we see that (1) holds follows from (3.3.2) and that (2) holds follows from (3.3.1).

Now suppose that conditions (1) and (2) hold. That (3.3.2) holds follows from (1), so we need only establish (3.3.2). Let then a,b and c be distinct elements of M. Let α be a meridian isomorphism from M onto $\mathfrak{W}$. Since $\mathcal{G}$ is 3-transitive, there exists an element $\gamma\in\mathcal{G}$ such that

$$\gamma(a) = \alpha^{-1}(\widetilde{\color{red}\blacksquare}),\quad \gamma(b) = \alpha^{-1}(\widetilde{\color{green}\blacksquare}) \quad\text{and}\quad \gamma(c) = \alpha^{-1}(\widetilde{\blacksquare}). \tag{3}$$

Letting $\phi\equiv\gamma^{-1}\circ\omega\circ\gamma$, we see from (3) that (3.3.1) holds. Q.E.D.

(6.11) Definition Let M be a set with at least four elements and let $\mathcal{G}$ be a meridian group of permutations of M. For any basis {u,v,w} of M the function

$$\mathcal{W}\ni[a,b,c,d]\hookrightarrow \widetilde{\boxed{\tfrac{u}{a,a}\ \tfrac{v}{b,b}\ \tfrac{w}{c,c}}}^{\,-1}(\langle a,b,c,d\rangle^{\sim})\in M \tag{1}$$

will be called a **cross ratio on** M.

(6.12) Theorem Let κ be a cross ratio on M, where M is a set of cardinality at least 4 and $\mathcal{G}$ is a meridian group of permutations of M. Let $\phi|M\hookrightarrow M$ be a bijection. Then the following two statements are equivalent:

$$\phi\in\mathcal{G}; \tag{1}$$

$$(\forall\,\{a,b,c,d\}\subset M\colon\ \langle a,b,c,d\rangle\in\mathcal{W})\quad \kappa([a,b,c,d]) = \kappa([\phi(a),\phi(b),\phi(c),\phi(d)]). \tag{2}$$

Proof. That (1) implies (2) follows directly from the definition of $\sim$.

Suppose that (2) holds. Let {u,v,w} be the basis of M such that

$$\kappa([a,b,c,d]) = \widetilde{\boxed{\tfrac{u}{a,a}\ \tfrac{v}{b,b}\ \tfrac{w}{c,c}}}^{\,-1}\circ\langle a,b,c,d\rangle \text{ for every M-quadriad } \{a,b,c,d\}. \tag{3}$$

Let t be any element of M. Then

$$\begin{aligned}
&<\text{u,v,w,t}> \sim \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}} \circ <\text{u,v,w,t}> = \\
&<\boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(u), \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(v), \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(w), \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(t)> = \\
&<\phi(u),\phi(v),\phi(w), \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(t)> .
\end{aligned} \tag{4}$$

From (4) follows that

$$<\text{u,v,w,t}>^{\sim} = <\phi(u),\phi(v),\phi(w), \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(t)>^{\sim} \tag{5}$$

On the other hand we have

$$\kappa([\phi(u),\phi(v),\phi(w),\phi(t)]) \overset{\text{by } (2)}{=\!=\!=} \kappa([u,v,w,t]) \quad \Longrightarrow \quad <\phi(u),\phi(v),\phi(w),\phi(t)>^{\sim} \overset{\text{by } (6.11)}{=\!=\!=} <\text{u,v,w,t}>^{\sim} \tag{6}$$

Let $a \equiv \phi(u)$, $b \equiv \phi(v)$ and $c \equiv \phi(w)$. From (5) and (6) we have

$$\begin{aligned}
&\widetilde{\boxed{\begin{matrix} a & b & c \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}\left(\boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(t)\right) = <\phi(u),\phi(v),\phi(w), \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(t)>^{\sim} = \\
&<\phi(u),\phi(v),\phi(w),\phi(t)>^{\sim} = \widetilde{\boxed{\begin{matrix} a & b & c \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(\phi(t)) .
\end{aligned}$$

Since $\widetilde{\boxed{\begin{matrix} a & b & c \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}$ is bijective, it follows that $\phi(t) = \boxed{\begin{matrix} u & v & w \\ \phi(u) & \phi(v) & \phi(w) \end{matrix}}(t)$. That is to say, ϕ is an element of $\mathcal{G}$. Q.E.D.

(6.13) Theorem Let $\mathcal{G}$ be a meridian family of permutations on a set M with cardinality at least 4. Let $\{0,1,\infty\}$ be a basis for M. Then

$$(\forall \{w,x,y,z\} \subset M\colon \#\{w,x,y,z\} \geq 3) \quad \widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}^{-1}(<y,x,w,z>^{\sim}) = \frac{(w\text{-}y)\cdot(x\text{-}z)}{(x\text{-}y)\cdot(w\text{-}z)} \tag{1}$$

where the binary operations are the field operations as in (2.5).

Proof. The function $\widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}$ is the projective mapping from M onto $\mathfrak{W}$ satisfying

$$\widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(0) = \widetilde{\blacksquare}, \quad \widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(1) = \widetilde{\color{red}\blacksquare} \quad \text{and} \quad \widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(\infty) = \widetilde{\color{green}\blacksquare}$$

while

$$\widetilde{\boxed{\begin{matrix} y & x & w \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(y) = \widetilde{\blacksquare}, \quad \widetilde{\boxed{\begin{matrix} y & x & w \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(x) = \widetilde{\color{red}\blacksquare} \quad \text{and} \quad \widetilde{\boxed{\begin{matrix} y & x & w \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}(w) = \widetilde{\color{green}\blacksquare}$$

whence follows that

$$\begin{aligned}
&\widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}^{-1}(<y,x,w,z>^{\sim})(w) = \infty, \quad \widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}^{-1}(<y,x,w,z>^{\sim})(x) = 0 \\
\text{and} \quad &\widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}^{-1}(<y,x,w,z>^{\sim})(y) = 1 .
\end{aligned} \tag{2}$$

In terms of the field operations, the element $\widetilde{\boxed{\begin{matrix} 1 & 0 & \infty \\ \text{a,a} & \text{b,b} & \text{c,c} \end{matrix}}}^{-1}(<y,x,w,z>^{\sim})$ becomes a linear fractional transformation. The linear fractional transformation satisfying (2) evidently is (1), viewed as a function of z. Q.E.D.

(6.14) Harmonic Wurfs Let $\mathcal{G}$ be a meridian family of permutations on a set M with cardinality at least 4. We know from (6.10) that $\tilde{\sigma}_{\color{green}\blacksquare}$ is an involution which fixes the point $\widetilde{\color{green}\blacksquare}$. From (2.11) follows that $\tilde{\sigma}_{\color{green}\blacksquare}$ has a second fixed point which we shall call $\widetilde{\color{green}\blacktriangledown}$:

$$\tilde{\sigma}_{\color{green}\blacksquare}(\widetilde{\color{green}\blacktriangledown}) = \widetilde{\color{green}\blacktriangledown}. \tag{1}$$

One can easily check that

$$\widetilde{\color{green}\blacktriangledown} = \{\langle a,b,c,d\rangle : \{\{a,d\},\{b,c\}\} \text{ is a harmonic pair.}\} \tag{2}$$

Furthermore, if ω is as in Theorem (6.10), then

$$\omega(\widetilde{\blacksquare}) = \widetilde{\color{green}\blacktriangledown} \quad \text{and} \quad \omega(\widetilde{\color{green}\blacktriangledown}) = \widetilde{\color{red}\blacksquare}. \tag{3}$$

Similarly, there are sets $\widetilde{\color{red}\blacktriangledown}$ and $\widetilde{\blacktriangledown}$ such that

$$\tilde{\sigma}_{\color{red}\blacksquare}(\widetilde{\color{red}\blacktriangledown}) = \widetilde{\color{red}\blacktriangledown} = \{\langle a,b,c,d\rangle : \{\{a,c\},\{b,d\}\} \text{ is a harmonic pair}\} \tag{4}$$

and

$$\tilde{\sigma}_{\blacksquare}(\widetilde{\blacktriangledown}) = \widetilde{\blacktriangledown} = \{\langle a,b,c,d\rangle : \{\{a,b\},\{c,d\}\} \text{ is a harmonic pair.}\} \tag{5}$$

(6.15) Remarks Let M be a meridian. Then $\mathfrak{W}$ is a meridian isomorphic with M which has three distinguished points: $\widetilde{\color{green}\blacksquare}$, $\widetilde{\blacksquare}$ and $\widetilde{\color{red}\blacksquare}$. In other words, $\mathfrak{W}$ comes equipped with a basis and so is a sort of intrinsic representation of M which "almost" has a distinguished basis, and so "almost" has a distinguished field.

(6.16) Example The set $\{\widetilde{\color{green}\blacktriangledown},\widetilde{\blacktriangledown},\widetilde{\color{red}\blacktriangledown}\}$ may not always have cardinality 3. It is not hard to show that the cardinality is 1 when it is not 3. With the sphere, circle (and line) meridians, it has cardinality 3. For the meridian which corresponds to the field {-1,0,1}, with ordinary multiplication and addition, it has cardinality 1.

7. Meridian Exponentials and Arcs

(7.1) Introduction It could be asserted with some justification that the circle or line meridian is the most useful meridian. That being so, it is reasonable to inquire into what distinguishes it from the others.

Suppose then that M is a circle meridian, and let $[0,1,\infty]$ be any ordered basis for M. We may regard $F \equiv M \triangle \{\infty\}$ as the field of real numbers where the binary operations are defined as in (2.5). There is a continuous function

$$\omega \big| F \hookrightarrow F$$

such that

$$(\forall \{m,n\} \subset \mathbb{N}) \quad \omega\left(\tfrac{m}{n}\right) = \frac{\overbrace{2 \cdot \ldots \cdot 2}^{\text{m times}}}{\underbrace{2 \cdot \ldots \cdot 2}_{\text{n times}}}. \tag{1}$$

This function thus has the property that

$$(\forall \{x,y\} \subset F) \quad \omega(x+y) = \omega(x) \cdot \omega(y), \tag{2}$$

and is commonly denoted by

$$(\forall x \in F) \quad 2^{x} \equiv \omega(x) \tag{3}$$

It turns out that that the existence of such an “exponential” function can be described in terms of the meridian structure and that the existence of such is in a certain sense sufficient, as well as necessary, to render a meridian isomorphic to a circle meridian.

(7.2) Review and Discussion Let $\mathcal{M}$ be any meridian family of involutions on a set M of cardinality of at least 4. For $\{a,d\} \subset M$, the family

$$\overset{a\leftrightarrow d}{\mathcal{M}} \equiv \{\phi \in \mathcal{M} : \phi(a) = d\} \tag{1}$$

is an abelian function libra by definition (2.3.3) and consequence (2.3.4). By (2.3.2) and by (4.14), $\overset{a\leftrightarrow d}{\mathcal{M}}$ is an inner involution libra on

$$M_{(a,d)} \equiv M \triangle \{a,d\}. \tag{2}$$

The function libra $\overset{a\leftrightarrow d}{\mathcal{M}}$ (*cf.* (4.19)) induces a libra operation on The set $M_{(a,d)}$:

$$(\forall \{x,y,z\} \subset M_{(a,d)}) \quad \lfloor x,y,z \rfloor \equiv \begin{cases} \boxed{x,x;z,z}(y) & \text{if } x \neq z; \\ y & \text{if } x = z. \end{cases} \tag{3}$$

and, for each $t \in M_{(a,d)}$,

$$\sigma_{(a,d;t)} \Big| \overset{a\leftrightarrow d}{\mathcal{M}} \ni \phi \hookrightarrow \phi(t) \in M_{(a,d)} \quad \text{is a libra isomorphism.} \tag{4}$$

When $a = d$, we shall use the abbreviation:

$$\sigma_{(d;t)} \equiv \sigma_{(d,d;t)}. \tag{5}$$

When $a \neq d$, it may happen that an element of $\overset{a\leftrightarrow d}{\mathcal{M}}$ has no fixed points. Therefore we introduce the notation

$$\overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare} \equiv \{\phi \in \overset{a\leftrightarrow d}{\mathcal{M}} : (\exists\, x \in M_{(a,d)}) \quad \phi(x) = x\}\,. \tag{6}$$

It follows from Theorem (2.15) that each element of $\overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$ fixes exactly two points. We recall that a **basis** for M is just a subset of M of cardinality 3, and that an **ordered basis** is an ordered triple of which the elements comprise a basis. For an ordered basis [a,l,d], we adopt the notation

$$M_{[a,l,d]} \equiv \{\phi(l) : \phi \in \overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}\}\,. \tag{7}$$

For $t \in M_{(a,d)}$, we shall write $\sigma_{(a,l,d;t)}$ for the restriction of $\sigma_{(a,d;t)}$ to $M_{[a,l,d]}$:

$$\sigma_{(a,l,d;t)} \Big| M_{[a,l,d]} \ni \phi \hookrightarrow \phi(t) \in M_{(a,d)}\,. \tag{8}$$

(7.3) Theorem Let [a,l,d] be an ordered basis for a meridian M, suppose that $\overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$ is balanced in $\overset{a\leftrightarrow d}{\mathcal{M}}$ and let m be an element of $M_{[a,l,d]}$. Then

$$M_{[a,l,d]} = M_{[a,m,d]}\,. \tag{1}$$

Proof. Let k be any element of $M_{[a,m,d]}$ and select $\psi \in \overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$ such that $\psi(m) = k$. Let $\phi \in \overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$ be such that $\phi(l) = m$. By hypothesis we have $\boxed{a,d;k,k} \circ \psi \circ \phi$ in $\overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$. Since

$$\boxed{a,d;k,k} \circ \psi \circ \phi(l) = \boxed{a,d;k,k} \circ \psi(m) = \boxed{a,d;k,k}(k) = k\,,$$

we have shown that

$$M_{[a,m,d]} \subset M_{[a,l,d]}\,. \tag{2}$$

Since $\phi(m) = l$, we have $l \in M_{[a,m,d]}$ and so, interchanging the roles of l and m, (2) becomes

$$M_{[a,l,d]} \subset M_{[a,m,d]}\,. \tag{3}$$

That (1) holds, follows from (2) and (3). Q.E.D.

(7.4) More Discussion Let [a,l,d] be an ordered basis for the meridian M and suppose that $\overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$ is balanced in the libra $\overset{a\leftrightarrow d}{\mathcal{M}}$. From (7.2.4) and (7.3) follows that $M_{[a,l,d]}$ is balanced in the libra $M_{(a,d)}$ and, for each $t \in M_{[a,l,d]}$,

$$\sigma_{(a,l,d;t)} \text{ is a libra isomorphism of } \overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare} \text{ onto } M_{[a,l,d]}\,. \tag{1}$$

It was shown in (4.15.5) that there can exist no libra isomorphism of $\overset{d\leftrightarrow d}{\mathcal{M}}$ onto $\overset{a\leftrightarrow d}{\mathcal{M}}$ if $a \neq d$. Nonetheless it does make sense to consider the possibility of an isomorphism of $\overset{d\leftrightarrow d}{\mathcal{M}}$ onto $\overset{a\leftrightarrow d}{\mathcal{M}}_{\blacksquare}$. If we return to the example of (7.1), the funtion $\omega \big| F \ni x \hookrightarrow 2^x \in F$ may also be expressed as

$$\omega \Big| M_{(\infty,\infty)} \ni x \hookrightarrow 2^x \in M_{[0,1,\infty]}\,. \tag{2}$$

We define

$$\Omega \Big| \overset{\infty\leftrightarrow\infty}{\mathcal{M}} \ni \boxed{\infty,\infty;0,c} \hookrightarrow \boxed{\infty,0;1,2^c} \in \overset{\infty\leftrightarrow 0}{\mathcal{M}}_{\blacksquare} \tag{3}$$

and observe that, for all $\{a,b,c\}\subset M_{(\infty,\infty)}$

$$\Omega(\llbracket \boxed{\infty,\infty;0,a},\boxed{\infty,\infty;0,b},\boxed{\infty,\infty;0,c}\rrbracket) = \Omega(\boxed{\infty,\infty;0,a}\circ\boxed{\infty,\infty;0,b}\circ\boxed{\infty,\infty;0,c}) =$$

$$\Omega(\boxed{\infty,\infty;0,a\text{-}b+c}) = \boxed{\infty,0;1,2^{a\text{-}b+c}} \overset{\text{by (2.5.6)}}{=\!=\!=} \boxed{\infty,0;1,2^{a}}\circ\boxed{\infty,0;1,2^{b}}\circ\boxed{\infty,0;1,2^{c}} =$$

$$\llbracket \boxed{\infty,0;1,2^{a}},\boxed{\infty,0;1,2^{b}},\boxed{\infty,0;1,2^{c}}\rrbracket = \llbracket \Omega(\boxed{\infty,\infty;0,a}),\Omega(\boxed{\infty,\infty;0,b}),\Omega(\boxed{\infty,\infty;0,c})\rrbracket$$

whence follows that Ω is a libra isomorphism of $\overset{\infty\leftrightarrow\infty}{\mathcal{M}}$ onto $\overset{\infty\leftrightarrow 0}{\mathcal{M}_{\blacksquare}}$. Furthermore

$$\omega = \sigma_{(0,1,\infty;1)}\circ\Omega\circ\sigma_{(\infty;0)}{}^{-1}. \tag{4}$$

We note that

$$1=\omega(0) \quad\Longrightarrow\quad \Omega(\boxed{\infty,\infty;0,0}) = \boxed{0,\infty;1,1} = \boxed{0,\infty;\omega(0),\omega(0)}, \tag{5}$$

that

$$\omega\circ\omega(0) = \omega(1) = 2 = \boxed{\infty,\infty;0,1}(1) \quad\Longrightarrow\quad \Omega\circ\Omega(\boxed{\infty,\infty;0,0}) = \boxed{\infty,0;0,\omega(0)}\circ\Omega(\boxed{\infty,\infty;0,0})\circ\boxed{\infty,0;0,\omega(0)} \tag{6}$$

and that

$$\vec{\omega}(M_{[0,\omega(0),\infty]}) = \vec{\omega}(M_{[0,1,\infty]}) = M_{[1,2,\infty]} = M_{[\omega(0),\omega\circ\omega(0),\infty]}. \tag{7}$$

Results (4), (5), (6) and (7) suggest the following definitions.

(7.5) Definitions Let M be a meridian and let $[a,l,d]$ be an ordered basis for M. Suppose that $\overset{a\leftrightarrow d}{\mathcal{M}_{\blacksquare}}$ is a balanced subset of the libra $\overset{a\leftrightarrow d}{\mathcal{M}}$ and that ω is a libra isomorphism from $M_{(d,d)}$ onto $M_{[a,l,d]}$. If there exists a libra isomorphism $\Omega\big|\overset{d\leftrightarrow d}{\mathcal{M}}\hookrightarrow\overset{a\leftrightarrow d}{\mathcal{M}_{\blacksquare}}$ such that

$$\omega = \sigma_{(a,l,d;\omega(a))}\circ\Omega\circ\sigma_{(d;a)}{}^{-1}, \tag{1}$$

we shall say that ω is a **meridian libra isomorphism**. If, in addition

$$\Omega(\boxed{d,d;a,a}) = \boxed{a,d;\omega(a),\omega(a)}, \tag{2}$$

$$\Omega\circ\Omega(\boxed{d,d;a,a}) = \boxed{d,a;a,\omega(a)}\circ\Omega(\boxed{d,d;a,a})\circ\boxed{d,a;a,\omega(a)} \tag{3}$$

and

$$\vec{\omega}(M_{[a,l,d]}) = M_{[\omega(a),\omega\circ\omega(a),d]} \tag{4}$$

we shall say that ω is a 2-**exponential isomorphism on** $M_{(d,d)}$ **originating at** a, and we say that Ω is the **progenitor of** ω. We note that (1) implies

$$\Omega = \sigma_{(a,l,d;\omega(a))}{}^{-1}\circ\omega\circ\sigma_{(d;a)}. \tag{5}$$

Furthermore $\omega(a)$ is in $M_{[a,l,d]}$ and so Theorem (7.3) implies that

$$M_{[a,\omega(a),d]} = M_{[a,l,d]}. \tag{6}$$

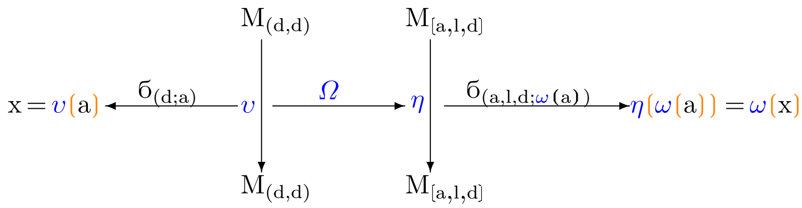
For a 2-exponential ω originating at a with progenitor Ω as above, we shall say that d is its **singular point** and that $\omega(a)$ is its **issue**. We note that the issue of ω is one of the two fixed points of $\Omega(\boxed{d,d;a,a})$.

If there exists a 2-exponential isomorphism on the meridian M as above, we shall say that M is an **exponential meridian**.

(7.6) Theorem Let M be an exponential meridian and let $\mathcal{M}$ be a meridian family of involutions on M. Then

$$(\forall\,[\mathrm{s,t,v}]\text{ an ordered basis for M})(\exists\,\psi\text{ a 2-exponential originating at s onto }\mathrm{M}_{[\mathrm{s,t,v}]})\,. \tag{1}$$

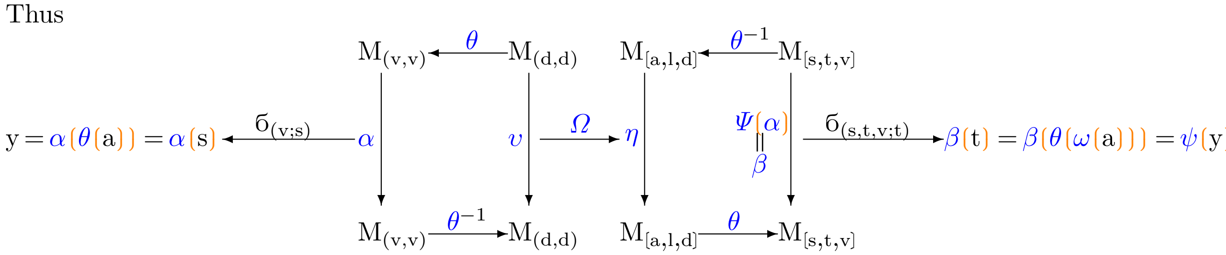
Proof. By hypothesis there exists an ordered basis $[\mathrm{a,l,d}]$ and a 2-exponential isomorphism ω with progenitor Ω and with base a from $\mathrm{M}_{(\mathrm{d,d})}$ onto $\mathrm{M}_{[\mathrm{a,l,d}]}$ such that (7.5.2), (7.5.3) and (7.5.4) hold. Thus



We define

$$\theta=\boxed{\begin{smallmatrix}\mathrm{a} & \omega(\mathrm{a}) & \mathrm{d}\\ \mathrm{s} & \mathrm{t} & \mathrm{v}\end{smallmatrix}}\,,\quad \Psi\Big|\overset{\mathrm{v}\leftrightarrow\mathrm{v}}{\mathcal{M}}\ni\phi\hookrightarrow\theta\circ\Omega(\theta^{-1}\circ\phi\circ\theta)\circ\theta^{-1}\in\overset{\mathrm{s}\leftrightarrow\mathrm{v}}{\mathcal{M}_{\blacksquare}}\quad\text{and}\quad\psi\equiv б_{(\mathrm{s,t,v;l})}\circ\Psi\circ б_{(\mathrm{v;s})}{}^{-1}\,. \tag{2}$$

Thus



Direct computations show that

$$\overset{\mathrm{v}\leftrightarrow\mathrm{v}}{\mathcal{M}}=\{\theta\circ\phi\circ\theta^{-1}:\phi\in\overset{\mathrm{d}\leftrightarrow\mathrm{d}}{\mathcal{M}}\}\quad\text{and}\quad\overset{\mathrm{s}\leftrightarrow\mathrm{v}}{\mathcal{M}}=\{\theta\circ\phi\circ\theta^{-1}:\phi\in\overset{\mathrm{a}\leftrightarrow\mathrm{d}}{\mathcal{M}}\}\,. \tag{3}$$

Direct computation, along with (7.5.2), (7.5.3), (7.5.4), (2) and (3) show that

$$\Psi(\boxed{\mathrm{v,v;s,s}})=\boxed{\mathrm{s,v;t,t}}\,, \tag{4}$$

$$\Psi\circ\Psi(\boxed{\mathrm{v,v;s,s}})=\boxed{\mathrm{v,s;s,}\psi(\mathrm{s})}\circ\Psi(\boxed{\mathrm{v,v;s,s}})\circ\boxed{\mathrm{v,s;s,}\psi(\mathrm{s})} \tag{5}$$

and

$$\overrightarrow{\psi}(\mathrm{M}_{[\mathrm{s,t,v}]})=\mathrm{M}_{[\psi(\mathrm{s}),\psi\circ\psi(\mathrm{s}),\mathrm{v}]} \tag{6}$$

That (1) holds now follows from (4), (5) and (6). Furthermore Ψ is the progenitor of ψ. Q.E.D.

(7.7) Theorem Let M be an exponential meridian. Then the underlying field has characteristic 0.

Proof. Let $\omega\big|\mathrm{M}_{(\infty,\infty)}\hookrightarrow\mathrm{M}_{[0,1,\infty]}$ be a 2-exponential function originating at 0 with issue 1. Assume that the underlying field were of characteristic p different from 0 (and 2 of course). Then p were a prime number different from 2. We introduce field operations for the basis $\{0,1,\infty\}$ of M. Choose $\mathrm{x}\in\overset{0\leftrightarrow\infty}{\mathcal{M}}$ such that

$$\omega(\mathrm{x})=\overbrace{1+1+\ldots+1}^{\text{p-1 times}}.$$

Then

$$\omega(\mathrm{x+x})=\omega(\mathrm{x})\cdot\omega(\mathrm{x})=\overbrace{1+1+\ldots+1}^{(\text{p}\cdot\text{p-2}\cdot\text{p-1})\text{ times}}=1=\omega(\mathrm{o})\quad\Longrightarrow\quad\mathrm{x+x=o}$$

which would imply that the field had characteristic 2, which is absurd. Q.E.D.

(7.8) Lemma Suppose that x is an element of a meridian M and that $\{\alpha,\beta\}\subset \overset{x\leftrightarrow x}{\mathcal{M}}$. Then

$$(\exists\,\gamma\in \overset{x\leftrightarrow x}{\mathcal{M}})\quad \alpha=\gamma\circ\beta\circ\gamma\,. \tag{1}$$

Proof. Since α is an involution with a fixed point x, there is another fixed point a of α. Similarly, there is a point b distinct from x which β fixes. Letting $\delta\equiv \boxed{x,x;a,b}\circ\beta\circ\boxed{x,x;a,b}$, we see that

$$\delta(x)=x=\alpha(x) \quad\text{and}\quad \delta(a)=a=\alpha(a)$$

Since these two involutions fix two common distinct points, it follows from (3.5) that they must be identical. Letting $\gamma\equiv\boxed{x,x;a,b}$, we see that γ is in $\overset{x\leftrightarrow x}{\mathcal{M}}$ and that (1) holds. Q.E.D.

(7.9) Theorem Let [a,t,b] be an ordered basis for the exponential meridian M and let

$$u\equiv\boxed{a,a;b,b}(t)\,. \tag{1}$$

Then

$$M_{[a,t,b]}\cap M_{[a,u,b]}=\emptyset \tag{2}$$

and

$$M=M_{[a,t,b]}\cup M_{[a,u,b]}\cup\{a,b\}\,. \tag{3}$$

Proof. Assume that u were in $M_{[a,t,b]}$. Then

$$(\exists\,\sigma\in\overset{a\leftrightarrow b}{\mathcal{M}}_{\blacksquare})\quad \sigma(t)=u. \tag{4}$$

We would have

$$\sigma\circ\boxed{a,b;t,t}\circ\sigma(u)\ \overset{\text{by (1)}}{=\!=}\ \sigma\circ\boxed{a,b;t,t}(t)=\sigma(t)\ \overset{\text{by (1)}}{=\!=}\ u\,.$$

Since $\sigma\circ\boxed{a,b;t,t}\circ\sigma$ and $\boxed{a,b;t,t}$ would agree at u, a and b, it would follow that

$$\sigma\circ\boxed{a,b;t,t}\circ\sigma=\boxed{a,b;t,t}\,. \tag{5}$$

By Theorem (7.6) there would exist a 2-exponential ω from $M_{(b,b)}$ with base a and issue t onto $M_{[a,t,b]}$. Let Ω be the progenitor for ω and let $\theta\in\overset{b\leftrightarrow b}{\mathcal{M}}$ be such that $\Omega(\theta)=\sigma$. We would have

$$\Omega(\theta)\circ\Omega(\boxed{b,b;a,a})\circ\Omega(\theta)\ \overset{\text{by (7.5.2)}}{=\!=}\ \Omega(\theta)\circ\boxed{b,b;t,t}\circ\Omega(\theta)=$$

$$\sigma\circ\boxed{a,b;t,t}\circ\sigma\ \overset{\text{by (5)}}{=\!=}\ \boxed{a,b;t,t}\ \overset{\text{by (7.5.2)}}{=\!=}\ \Omega(\boxed{b,b;a,a})$$

which would imply that

$$\theta\circ\boxed{b,b;a,a}\circ\theta=\boxed{b,b;a,a}\,. \tag{6}$$

The only way equation (6) could hold in the context of $\overset{b\leftrightarrow b}{\mathcal{M}}$ would be for θ to be $\boxed{b,b;a,a}$, which, in view of (7.5.2) and the fact that t is the issue of ω, would imply that $\sigma=\boxed{a,b;t,t}$. Consequently

$$\boxed{b,b;a,a}(t)=u=\sigma(t)=\boxed{a,b;t,t}(t)=t \quad\Longrightarrow\quad t\in\{b,a\}$$

which would be absurd: it follows that

$$u\notin M_{[a,t,b]}\,. \tag{7}$$

Now assume that there existed an element $x\in M_{[a,t,b]}\cap M_{[a,u,b]}$. Then

$$(\exists\, \{\phi,\eta\} \subset \overset{a\leftrightarrow b}{\mathcal{M}}_{\blacksquare}) \quad \phi(t) = x = \eta(u)\,.$$

The meridian M being an exponential meridian, $\overset{a\leftrightarrow b}{\mathcal{M}}_{\blacksquare}$ is a libra and so we have $\eta \circ \phi \circ \boxed{a,b;t,t} \in \overset{a\leftrightarrow b}{\mathcal{M}}_{\blacksquare}$. Consequently

$$M_{[a,t,b]} \ni \eta \circ \phi \circ \boxed{a,b;t,t}(t) = \eta \circ \phi(t) = \eta(x) = u$$

which violates (7). This establishes (2).

Let p be any element of $M_{(a,b)}$. Let q be the fixed point of $\boxed{a,b;p,p}$ which is distinct from p. Evidently $\boxed{a,b;t,t} \circ \boxed{a,a;b,b} \circ \boxed{a,b;t,t}$ is in $\mathcal{M}$ and leaves both a and b fixed: hence

$$\boxed{a,b;t,t} \circ \boxed{a,a;b,b} \circ \boxed{a,b;t,t} = \boxed{a,a;b,b}\,.$$

It follows that

$$\boxed{a,b;t,t} \circ \boxed{a,a;b,b} = \boxed{a,a;b,b} \circ \boxed{a,b;t,t} \quad \Longrightarrow \quad \boxed{a,b;t,t} \circ \boxed{a,a;b,b}(t) = \boxed{a,a;b,b} \circ \boxed{a,b;t,t}(t) = \boxed{a,a;b,b}(t) \tag{8}$$

Since $\overset{a\leftrightarrow b}{\mathcal{M}}_{\blacksquare}$ is isomorphic as a libra to $\overset{b\leftrightarrow b}{\mathcal{M}}$, it follows from Lemma (7.8) that there exists $\upsilon \in \overset{a\leftrightarrow b}{\mathcal{M}}_{\blacksquare}$ such that

$$\boxed{a,b;p,p} = \upsilon \circ \boxed{a,b;t,t} \circ \upsilon$$

which implies both

$$\boxed{a,b;p,p}(\upsilon(t)) = \upsilon \circ \boxed{a,b;t,t} \circ \upsilon(\upsilon(t)) = \upsilon(t) \tag{9}$$

and

$$\begin{aligned} \boxed{a,b;p,p}(\upsilon(\boxed{b,b;a,a}(t))) &= \upsilon \circ \boxed{a,b;t,t} \circ \upsilon(\upsilon(\boxed{b,b;a,a}(t))) = \\ \upsilon \circ \boxed{a,b;t,t} \circ \boxed{b,b;a,a}(t) &\overset{\text{by (8)}}{=\!=\!=} \upsilon(\boxed{a,a;b,b}(t))\,. \end{aligned} \tag{10}$$

By definition of p and q it now follows from (9) and (10) that

$$\{\upsilon(t)\,, \upsilon(\boxed{a,a;b,b}(t))\} = \{p,q\}\,.$$

If $\upsilon(t) = p$, then $p \in M_{[a,t,b]}$. If $\upsilon(\boxed{a,a;b,b}(t)) = p$, then $p \in M_{[a,u,b]}$. This establishes (3). Q.E.D.

(7.10) Definitions and Notation Let b be any ordered basis for M: that is, an ordered triple of pairwise distinct elements. We shall adopt the following notation:

$$[\overset{b}{0}, \overset{b}{1}, \overset{b}{\infty}] \equiv b\,. \tag{1}$$

The field operations relative to b will be denoted by $\overset{b}{+}$, $\overset{b}{-}$, $\overset{b}{\cdot}$, and $-_b$: thus

$$x \overset{b}{+} y = \boxed{\begin{matrix} & \overset{b}{\infty} & \\ x & \overset{b}{0} & y \\ & \overset{b}{\infty} & \end{matrix}}\,, \quad x \overset{b}{-} y = \boxed{\begin{matrix} & \overset{b}{\infty} & \\ x & y & \overset{b}{0} \\ & \overset{b}{\infty} & \end{matrix}} \quad x \overset{b}{\cdot} y = \boxed{\begin{matrix} & \overset{b}{\infty} & \\ x & \overset{b}{1} & y \\ & \overset{b}{0} & \end{matrix}} \quad \text{and} \quad \frac{x}{y}_b = \boxed{\begin{matrix} & \overset{b}{\infty} & \\ x & y & \overset{b}{1} \\ & \overset{b}{0} & \end{matrix}}\,. \tag{2}$$

We shall denote by

$$F_b \tag{3}$$

the corresponding field $M_{(\infty,\infty)}$.

When it is clear from the context which ordered basis b is under consideration, we shall occasionally suppress the symbol "b" from the field operations, shall occasionally suppress the symbol "$\overset{b}{\cdot}$" completely, and shall occasionally introduce the exponential notation

$$x^n \;=\; \overbrace{x{\cdot}x{\cdot}\ldots{\cdot}x}^{\text{n times}}. \tag{4}$$

(7.11) Lemma Let $M_{[0,1,\infty]}$ be an ordered basis for a meridian M. Let $2\equiv 1{+}1$. Then

$$M_{[1,2,\infty]} = \{1{+}t^2 : t\in M_{(\infty,\infty)}\}\,. \tag{1}$$

Proof. Let x be an element of $M_{[1,2,\infty]}$. Then there exists $\theta\in \overset{\infty\leftrightarrow 1}{\mathcal{M}_{\blacksquare}}$ such that $\theta(2) = x$. Since $\theta(1) = \infty$, there exists $b\in M_{(\infty,\infty)}$ such that

$$(\forall\, s\in M_{(\infty,\infty)})\quad \theta(s) = \frac{s+b}{s-1}\,.$$

Since θ has a fixed point p, it follows from the quadratic formula that $b{+}1$ has a square root t. Thus

$$x = \theta(2) = \frac{2+b}{2-1} = 1{+}b{+}1 = 1{+}t^2\,.$$

Now suppose that $x = 1{+}t^2$ for some t. Setting $b\equiv 1{-}t^2$, direct calculation shows that the function

$$F\ni s \hookrightarrow \frac{s-b}{s-1}\in M \quad \text{is in } \overset{\infty\leftrightarrow 1}{\mathcal{M}_{\blacksquare}} \text{ and sends 2 to } x\,,$$

whence follows that x is in $M_{[1,2,\infty]}$. Q.E.D.

(7.12) Theorem Let $[a,t,b]$ be an ordered basis for an exponential meridian M and let ϕ be an element of $\overset{a\leftrightarrow b}{\mathcal{M}_{\blacksquare}}$. Then one fixed point of ϕ is in $M_{[a,t,b]}$ and one is not.

Proof. Since $\phi\circ\boxed{a,a;b,b}\circ\phi$ and $\boxed{a,a;b,b}$ both fix the distinct points a and b, it follows that they are the same. Let p and q be the fixed points of ϕ. Then

$$\phi(\boxed{a,a;b,b}(p)) = \phi(\phi\circ\boxed{a,a;b,b}\circ\phi(p)) = \boxed{a,a;b,b}(p)\,.$$

Since p is neither a nor b, it follows that $\boxed{a,a;b,b}(p) \neq p$ and so

$$\boxed{a,a;b,b}(p) = q\,. \tag{1}$$

From (7.9.4) follows that p is in either $M_{[a,t,b]}$ or in $M_{[a,\boxed{a,a;b,b}(t),b]}$. By Theorem (7.3) $M_{[a,p,b]}$ is either $M_{[a,t,b]}$ or $M_{[a,\boxed{a,a;b,b}(t),b]}$. By (1) and (7.9.3), whichever one $M_{[a,p,b]}$ is, $M_{[a,q,b]}$ is the other. Q.E.D.

(7.13) Lemma Let $M_{[0,1,\infty]}$ be an ordered basis for a meridian M. Then

$$M_{[0,1,\infty]} = \{f^2 : t\in M_{(\infty,\infty)}\} = \{f^2 : t\in M_{[0,1,\infty]}\}\,. \tag{1}$$

Proof. Let t be an element of $M_{(\infty,\infty)}$. Then $\boxed{0,t^2,1,0}$ is in $\overset{0\leftrightarrow\infty}{\mathcal{M}}$, leaves t fixed, and

$$\boxed{0,t^2,1,0}(1) = t^2 \quad\Longrightarrow\quad t^2\in M_{[0,1,\infty]}\,. \tag{2}$$

Now let x be an element of $M_{[0,1,\infty]}$. There exists $\phi\in\overset{0\leftrightarrow\infty}{\mathcal{M}_{\blacksquare}}$ such that $\phi(1) = x$. It follows from (7.12) that one of the fixed points of ϕ is in $M_{[0,1,\infty]}$: we shall denote that point by t. We have

$$\phi = \boxed{0,t^2,1,0} \quad\Longrightarrow\quad x = \phi(1) = \frac{t^2}{1} \quad\Longrightarrow\quad x = t^2 \tag{3}$$

That (1) holds is now a consequence of (2) and (3).

(7.14) Lemma Let M be an exponential meridian and $\mathcal{M}$ the meridian family of involutions on M. Let $[0,1,\infty]$ be an ordered basis for M and let "+" and "·" be the associated binary operations on the field $M_{(\infty,\infty)}$. Then,

$$(\forall\, \{x,y\} \subset M_{[0,1,\infty]}) \quad \{x{\cdot}y\,,\ \frac{y}{x},\ \ x{+}y\} \subset M_{[0,1,\infty]}\,. \tag{1}$$

Proof. From (7.13) follows that there exists $\{s,t\} \subset M_{[0,1,\infty]}$ such that

$$x = s^2 \quad \text{and} \quad y = t^2\,. \tag{2}$$

In particular

$$xy = (st)^2 \ \xrightarrow{\text{by (7.13)}}\ xy \in M_{[0,1,\infty]} \tag{3}$$

and

$$\frac{y}{x} = (\frac{t}{s})^2 \ \xrightarrow{\text{by (7.13)}}\ \frac{y}{x} \in M_{[0,1,\infty]}\,. \tag{4}$$

By (7.6) there exists a 2-exponential ω of $M_{(\infty,\infty)}$ onto $M_{[0,1,\infty]}$ with base 0 and issue 1. We have by (7.5.4)

$$M_{[1,2,\infty]} = \overrightarrow{\omega}(M_{[0,1,\infty]}) \subset \overline{\omega} = M_{[0,1,\infty]} \ \xrightarrow{\text{by (7.11.1)}}\ 1{+}\frac{y}{x} \in M_{[0,1,\infty]} \ \xrightarrow{\text{by (3)}}\ x(1{+}\frac{y}{x}) \in M_{[0,1,\infty]}\,.$$

This, with (3) and (4), implies (1). Q.E.D.

(7.15) Notation Let $b = [0,1,\infty]$ be a basis for a meridian M. For $\{a,b,c,d\} \subset F_b = M_{(\infty,\infty)}$, we adopt the notation

$$\overset{b}{\mathfrak{det}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv ad - bc\,. \tag{1}$$

Furthermore we denote

$$\mathcal{M}_\square \ \equiv\ \{\phi \in \mathcal{M} : \phi \text{ has no fixed point}\}\,. \tag{2}$$

(7.16) Theorem Let $[0,1,\infty]$ be a basis for an exponential meridian M. Let ϕ be an element of $\mathcal{M}$ and suppose that $\phi = \boxed{a,b,c,\text{-}a}$ as in (2.4). Then

$$\phi \in \mathcal{M}_\square \iff \overset{b}{\mathfrak{det}}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_{[0,1,\infty]}\,. \tag{1}$$

Proof. It follows from the quadratic equation that $\boxed{a,b,c,\text{-}a}$ has a fixed point if, and only if, $a^2{+}b{\cdot}c$ has a square root in $M_{(\infty,\infty)}$, which, in view of Lemma (7.13), is true if, and only if, $a^2{+}b{\cdot}c$ is in $M_{[0,1,\infty]}$. It follows from Theorem (7.9) that $a^2{+}b{\cdot}c$ is in $M_{[0,1,\infty]}$ if, and only if, $-a^2{-}b{\cdot}c$ is not in $M_{[0,1,\infty]}$. But

$$-a{\cdot}a - b{\cdot}c \ = \ \overset{b}{\mathfrak{det}}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

from which (1) holds. Q.E.D.

(7.17) Lemma Let M be an exponential meridian and let τ be a translation in $[\![\mathcal{M}]\!]$.[28] Then τ is the composition of nine elements of $\mathcal{M}_\square$.

[28] That is, τ has exactly one fixed point in M.

Proof. Let b be the ordered basis $[0,1,\infty]$ where ∞ is the fixed point of τ, 0 is any other point of M and $1 \equiv \tau^{-1}(0)$. Then

$$\tau = \boxed{1,-1,0,1}$$

and direct calculation[29] yields

$$\tau = \boxed{0,-6,1,0} \circ \boxed{0,-5,1,0} \circ \boxed{1,-1,2,-1} \circ \boxed{4,-5,5,-4} \circ \boxed{0,-4,1,0} \circ \boxed{0,-1,1,0} \circ \boxed{1,-1,2,-1} \circ \boxed{0,-5,1,0} \circ \boxed{0,-6,1,0} . \quad (1)$$

That each of the factors in (1) is in $\mathcal{M}_\square$ follows from (7.16). Q.E.D.

(7.18) Lemma Let $[0,1,\infty]$ be an ordered basis for an exponential meridian. Then

$$M_{[0,1,\infty]} \cap M_{[\infty,0,1]} \cap M_{[1,\infty,0]} = \emptyset \quad (1)$$

Proof. Assume that there existed some p in $M_{[0,1,\infty]} \cap M_{[\infty,0,1]} \cap M_{[1,\infty,0]}$. Then

$$(\exists\, [\alpha,\beta,\gamma] \in \overset{0\leftrightarrow\infty}{\mathcal{M}_\blacksquare} \times \overset{0\leftrightarrow 1}{\mathcal{M}_\blacksquare} \times \overset{1\leftrightarrow\infty}{\mathcal{M}_\blacksquare}) \quad \alpha(1) = \beta(\infty) = \gamma(0) = p\,.$$

It follows that

$$\alpha = \boxed{0,p,1,0}\,, \quad \beta = \boxed{p,\text{-}p,1,\text{-}p} \quad \text{and} \quad \gamma = \boxed{\text{-}1,p,\text{-}1,1}\,.$$

Applying the quadratic equation to each of the homographies α , β and γ, respectively, we obtain the formulae

$$\pm\sqrt{p}, \quad p \pm \sqrt{p\cdot p - p} \quad \text{and} \quad 1 \pm \sqrt{1-p}$$

respectively, for their fixed points. Since these three homographies would have fixed points, there would exist $\{a,b,c\} \in M_{(\infty,\infty)}$ such that

$$a^2 = p, \quad b\cdot b = p\cdot p - p \quad \text{and} \quad c\cdot c = 1 - p\,.$$

It would follow that

$$(a\cdot c)\cdot(a\cdot c) = -b\cdot b$$

which by (7.9.1) is impossible. Q.E.D.

(7.19) Theorem Let $[0,1,\infty]$ be a basis for an exponential meridian M. Then

$$\{\{0\},\{1\},\{\infty\},M_{[0,\boxed{0,0;\infty,\infty}(1),\infty]},M_{[1,\boxed{1,1;\infty,\infty}(0),\infty]},M_{[1,\boxed{1,1;0,0}(\infty),0]}\} \text{ is a partition of M.} \quad (1)$$

Proof. We first show that

$$M = (\{\{0\},\{1\},\{\infty\},M_{[0,\boxed{0,0;\infty,\infty}(1),\infty]},M_{[1,\boxed{1,1;\infty,\infty}(0),\infty]},M_{[1,\boxed{1,1;0,0}(\infty),0]}\}) \quad (2)$$

Let t be any element of $M \triangle \{0,1,\infty\}$. If $t \notin M_{[0,\boxed{0,0;\infty,\infty}(1),\infty]}$, then (7.9) implies that $t \in M_{[0,1,\infty]}$. Similarly, if t is not in $M_{[1,\boxed{1,1;\infty,\infty}(0),\infty]}$, then t is in $M_{[\infty,0,1]}$, and if t is not in $M_{[1,\boxed{1,1;0,0}(\infty),0]}$, then t is in $M_{[1,\infty,0]}$. It follows from (7.18) that these three things cannot occur concurrently. We have established (2).

Assume that the constituents of (1) were not pairwise disjoint. Without loss of generality, we may assume that $M_{[0,\boxed{0,0;\infty,\infty}(1),\infty]}$ and $M_{[1,\boxed{1,1;\infty,\infty}(0),\infty]}$ had a point x in common: which is to say that

$$x \in M_{[0,\text{-}1,\infty]} \cap M_{[1,2,\infty]}\,. \quad (3)$$

[29] One constructs the corresponding 2×2 matrices and uses matrix multiplication.

Let ω be a 2-exponential from $M_{(\infty,\infty)}$ to $M_{[0,1,\infty]}$. From (7.5.4) follows that there would exist $u \in M_{[0,1,\infty]}$ such that $\omega(u) = x$. But $\omega(u)$ is in $M_{[0,1,\infty]}$, which by (7.9.2) would contradict (3). Q.E.D.

(7.20) Corollary Let $b = [0,1,\infty]$ be an ordered basis for an exponential meridian M. Then

$$\{M_{[0,\frac{1}{2},1]},\{1\},M_{[1,2,\infty]}\} \text{ is a partition of } M_{[0,1,\infty]}\,. \tag{1}$$

Proof. From (7.19) follows that

$$\{\{0\},\{1\},\{\infty\},M_{[0,-1,\infty]},M_{[1,2,\infty]},M_{[0,\frac{1}{2},1]}\} \text{ is a partition of M} \tag{2}$$

and from (7.9) follows that

$$\{\{0\},\{\infty\},M_{[0,1,\infty]},M_{[0,-1,\infty]}\} \text{ is also a partition of M}\,. \tag{3}$$

That (1) holds follows from (2) and (3). Q.E.D.

(7.21) Lemma Let M be an exponential meridian and let $\rho \in \llbracket \mathcal{M} \rrbracket \triangle \mathcal{M}$ be a pure rotation.[30] Then there exists $\pi \in \mathcal{M}_\square$ and a translation τ such that

$$\rho = \pi \circ \tau\,. \tag{1}$$

Proof. Let ∞ be any element of M. Let $a \equiv \rho(\infty)$. Since ρ is not in $\mathcal{M}$, it follows from Corollary (2.11) that $\rho(a) \neq \infty$. Let $1 \equiv \rho^{-1}(\infty)$. It follows from (2.17) and (2.19) that there exists $0 \in M$ such that $\{\{0,\infty\},\{a,1\}\}$ is a harmonic pair. Introducing the canonical field operations on $F \equiv M_{(\infty,\infty)}$, we see that $a = -1$ and so, if $b \equiv \rho(0)$, we have

$$\rho = \boxed{1,b,-1,1}\,. \tag{2}$$

Thus the equation for ρ to have a fixed point x is

$$x^2 + b = 0\,.$$

Since ρ, being a rotation, has no fixed point, we have $b \in M_{[0,1,\infty]}$. From (7.11) follows that $1+b$ is in $M_{[1,2,\infty]}$. Since M is an exponential meridian, it follows from (7.14.1) that $1+b$ is in $M_{[0,1,\infty]}$, whence follows by Lemma (7.13) that there exists $m \in M_{[0,1,\infty]}$ such that

$$m^2 = 1+b\,. \tag{3}$$

Let

$$p \equiv \frac{b+2+2m}{b^2}\,. \tag{4}$$

We have

$$\begin{aligned} p^2b^2 - 2pb + 1 - 4p &= \Big(\frac{b+2+2m}{b^2}\Big)^2 b^2 - 2\Big(\frac{b+2+2m}{b^2}\Big)b + 1 - 4\Big(\frac{b+2+2m}{b^2}\Big) = \\ \frac{b^2+4+4m^2+4b+4m+2bm-2b^2-4b-2bm+b^2-4b-8-4m}{b^2} &= \frac{-4+4m^2-4b}{b^2} \\ \overset{\text{by (3)}}{=\!=\!=} \frac{-4+4(1+b)-4b}{b^2} &= 0 \end{aligned} \tag{5}$$

Since the left hand side of (5) is the discriminant of the quadratic polynomial

[30] That is, ρ has no fixed point.

$$\mathrm{px}^2+(\mathrm{pb}-1)\mathrm{x}+1\,, \tag{6}$$

it follows from (5) that that polynomial has a single root. This in turn implies that the homography $\boxed{\text{-1,1,-p,-pb}}$ has exactly one fixed point, and thus is a translation. Letting

$$\tau \equiv \boxed{\text{-1,1,-p,-pb}} \quad \text{and} \quad \pi \equiv \boxed{\text{0,1,-p,0}}\,,$$

we see that from (2) that (1) holds.

It remains to show that π has no fixed point. But the fixed point equation for π is

$$1 = \text{-px}^2$$

and, since 1 and px^2 are both in $\mathrm{M}_{[0,1,\infty]}$, it follows from (7.9.2) that this equation has no solution. Q.E.D.

(7.22) Theorem Let M be an exponential meridian and ϕ an element of $\llbracket\mathcal{M}\rrbracket$. Let $\mathrm{b} = [0,1,\infty]$ be any ordered basis for M and suppose that $\phi = \boxed{\text{a,b,c,d}}$ relative to that basis. Then

$$\overset{\mathrm{b}}{\mathfrak{Det}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{[0,1,\infty]} \quad \Longleftrightarrow \quad \text{f is a composition of involutions without fixed points.} \tag{1}$$

Proof. $[\Longleftarrow]$ If ϕ is a composition of involutions without fixed points, then it follows from (7.16) that the matrix of each component has a determinant in $\mathrm{M}_{[0,1,\infty]}$. It follows from Lemma (7.14) that the matrix of ϕ would have determinant in $\mathrm{M}_{[0,1,\infty]}$.

$[\Longrightarrow]$ Let then b be an ordered basis for M and suppose that the determinant of the matrix of $\phi = \boxed{\text{a,b,c,d}}$ is in $\mathrm{M}_{[0,1,\infty]}$. If ϕ is an involution, then it follows from Theorem (7.16) that the determinant of the matrix is in $\mathrm{M}_{[0,1,\infty]}$. If ϕ is a translation, then it follows from (7.17) that it is the composition of an involution without fixed elements of $\mathcal{M}_\square$, and so its matrix has determinant in $\mathrm{M}_{[0,1,\infty]}$. If ϕ is a pure rotation, then Lemma (7.21) implies that it is the composition of an involution without fixed points and a translation. Since the determinants of the matrices of both these constituents are in $\mathrm{M}_{[0,1,\infty]}$, it follows that the matrix of the rotation is as well.

Thus we may presume that ϕ is a non-involutive dilation and that the determinate of the matrix of $\boxed{\text{a,b,c,d}}$ is in $\mathrm{M}_{[0,1,\infty]}$. Let θ be any element of $\llbracket\mathcal{M}\rrbracket$ which sends 0to one of the fixed points of ϕ and ∞ to the other. The $\theta\circ\phi\circ\theta^{-1}$ is a non-involutive dilation and so has a matrix relative to b of the form $\boxed{\text{r,0,0,s}}$. Since $\mathfrak{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$is in $\mathrm{M}_{[0,1,\infty]}$, it follows that $\mathrm{rs} = \mathfrak{Det}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$is as well. Thus we can choose r to be 1 and $\mathrm{m} \in \mathrm{M}_{[0,1,\infty]}$ such $\mathrm{m}^2 = \mathrm{s}$. Thus

$$\boxed{\text{r,0,0,s}} = \boxed{\text{0,1,-m,0}} \circ \boxed{\text{0,-m,1,0}}\,. \tag{2}$$

Evidently $\mathfrak{Det}\begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}$and $\mathfrak{Det}\begin{pmatrix} 0 & 1 \\ -m & 0 \end{pmatrix}$are in $\mathrm{M}_{[0,1,\infty]}$, whence follows that the determinants of matrices of $\theta^{-1}\circ\boxed{\text{0,-m,1,0}}\circ\theta$ and $\theta^{-1}\circ\boxed{\text{0,1,-m,0}}\circ\theta$ are as well. Since

$$\phi = \theta^{-1}\circ\theta\circ\phi\circ\theta^{-1}\circ\theta = \theta^{-1}\circ\boxed{\text{0,1,-m,0}}\circ\boxed{\text{0,-m,1,0}}\circ\theta = (\theta^{-1}\circ\boxed{\text{0,1,-m,0}}\circ\theta)\circ(\theta^{-1}\circ\boxed{\text{0,-m,1,0}}\circ\theta)$$

and so ϕ is the composition of two involutions without fixed points. Q.E.D.

(7.23) Definition Recall that $\mathcal{G}$ is the notation for $\llbracket\mathcal{M}\rrbracket$: the group of homographies of M. We shall write

$$\mathcal{G}^+ \tag{1}$$

for the set of elements of $\mathcal{G}$ which are compositions of involutions without fixed points. This terminology is motivated by the characterization (7.22). In view of (7.22), $\mathcal{G}^+$ is a normal subgroup of $\mathcal{G}$.[31]

We define an equivalence relation $\approx$ on the family of all ordered bases of M by, for any two such $[a,b,c]$ and $[r,s,t]$,

$$[a,b,c] \approx [r,s,t] \quad \Longleftrightarrow \quad (\exists\, \phi \in \mathcal{G}^+ \,) \;\; [r,s,t] = [\phi(a), \phi(b), \phi(c)]\,. \tag{2}$$

There are two equivalence classes and we shall denote this set of two elements by

$$\{\circlearrowright, \circlearrowleft .\} \tag{3}$$

These equivalence classes will be called **orientations**.

Let $\circlearrowright$ be an orientation of an exponential meridian M. Let a and b be distinct elements of M. We write

$$\overset{\frown}{a,b} \;\; \equiv \;\; \{t \in M_{(a,b)} : [a,t,b] \in \circlearrowright\} \tag{4}$$

and

$$\overset{\sqcap}{a,b} \;\; \equiv \;\; \{a,b\} \cup \overset{\frown}{a,b}\;. \tag{5}$$

These two sets $\overset{\frown}{a,b}$ and $\overset{\sqcap}{a,b}$, respectively, will be called the $\circlearrowright$-**open arc associated with** $[a,b]$ and the $\circlearrowright$-**closed arc associated with** $[a,b]$, respectively. Thus, for $t \in \overset{\frown}{a,b}$, we have

$$M_{[a,t,b]} = \overset{\frown}{a,b} \quad \text{and} \quad \{a,b\} \cup M_{[a,t,b]} = \overset{\sqcap}{a,b}\;. \tag{6}$$

When the orientation $\circlearrowright$ is evident from the context, we shall sometimes omit the symbol $\circlearrowright$ from the terms $\circlearrowright$-open and $\circlearrowright$-closed.

(7.24) Proposition In an exponential meridian M

$$\text{translations are in } \mathcal{G}^+, \tag{1}$$

$$\text{pure rotations are in } \mathcal{G}^+, \tag{2}$$

$$\text{involutions with fixed points are not in } \mathcal{G}^+, \tag{3}$$

$$(\forall\, [a,b,t] \text{ an ordered basis})(\forall\, u \in M_{[a,b,t]}) \quad [a,b,t] \text{ and } [a,b,u] \text{ have the same orientation,} \tag{4}$$

$$(\forall\, [a,b,t] \text{ an ordered basis})(\forall\, u \notin M_{[a,t,b]}) \quad [a,t,b] \text{ and } [a,u,b] \text{ have opposite orientations,} \tag{5}$$

$$(\forall\, [a,t,b] \text{ an ordered basis}) \quad [a,b,t] \text{ and } [b,a,t] \text{ have opposite orientations} \tag{6}$$

and

$$(\forall\, [a,t,b] \text{ an ordered basis}) \quad [a,b,t] \text{ and } [b,t,a] \text{ have the same orientation.} \tag{7}$$

Proof. $[\Longrightarrow(1)]$ Follows from Lemma (7.17).

$[\Longrightarrow(2)]$ Follows from Lemma (7.21).

$[\Longrightarrow(3)]$ Follows from Theorem (7.16).

$[\Longrightarrow(4)]$ The involutions $\boxed{a,b;t,u}$ and $\boxed{a,b;u,u}$ both have fixed points and so the determinants of their matrices are not positive. It follows that the matrix of their composition is positive and so $\boxed{a,b;u,u} \circ \boxed{a,b;t,u}$ is in $\in \mathcal{G}^+$. Furthermore $\boxed{a,b;u,u} \circ \boxed{a,b;t,u} \circ [a,b,t] = [a,b,u]$.

$[\Longrightarrow(5)]$ The involution $\boxed{a,b;t,u}$ has no fixed point by Theorem (7.9) and $[b,a,t] = \boxed{a,b;t,u} \circ [a,b,t]$.

[31] A subgroup N of a group G be **normal** if, for each $x \in G$ and $n \in N$, $x \cdot n] \cdot x^{-1} \in N$.

[$\Longrightarrow$(6)] The involution $\boxed{\mathrm{a,b;t,t}}$ has a fixed point and $[\mathrm{b,a,t}] = \boxed{\mathrm{a,b;t,t}} \circ [\mathrm{a,b,t}]$.

[$\Longrightarrow$(7)] The involutions $\boxed{\mathrm{a,b;t,t}}$ and $\boxed{\mathrm{a,t;b,b}}$ have fixed points and so the determinants of their matrices are not positive. It follows that the matrix of their composition is positive and so $\boxed{\mathrm{a,t;b,b}} \circ \boxed{\mathrm{a,b;t,t}}$ is in $\in \mathcal{G}^{+}$. Furthermore $\boxed{\mathrm{a,t;b,b}} \circ \boxed{\mathrm{a,b;t,t}} \circ [\mathrm{a,b,t}] = [\mathrm{b,t,a}]$. Q.E.D.

(7.25) Corollary Let M be an exponential meridian with an orientation ⟳ and let a and b be distinct elements of M. Then

$$\{\{a\},\{b\}, \overset{\frown}{\mathrm{a,b}}, \overset{\frown}{\mathrm{b,a}}\} \text{ is a partition of } M, \tag{1}$$

$$(\forall\, t \in \overset{\frown}{\mathrm{a,b}}) \quad \{\overset{\frown}{\mathrm{a,t}}, \{t\}, \overset{\frown}{\mathrm{t,b}}\} \text{ is a partition of } \overset{\frown}{\mathrm{a,b}} \tag{2}$$

and

$$(\forall\, t \in \overset{\frown}{\mathrm{b,a}}) \quad \{\overset{\frown}{\mathrm{b,t}}, \{t\}, \overset{\frown}{\mathrm{t,a}}\} \text{ is a partition of } \overset{\frown}{\mathrm{b,a}}\,. \tag{3}$$

Proof. This follows from (7.20) and (7.24). Q.E.D.

(7.26) Theorem Let {a,b} and {p,q} be subsets of a circular meridian M, both of cardinality 2. Then

$$\text{if } p = a \text{ or } q = b, \text{ either } \quad \overset{\frown}{\mathrm{a,b}} \subset \overset{\frown}{\mathrm{p,q}} \quad \text{or} \quad \overset{\frown}{\mathrm{p,q}} \subset \overset{\frown}{\mathrm{a,b}}\,; \tag{1}$$

$$\text{if } p = b, \text{ then } \quad (\overset{\frown}{\mathrm{a,b}} \cap \overset{\frown}{\mathrm{p,q}}) \in \{\emptyset, \overset{\frown}{\mathrm{a,q}}\}\,; \tag{2}$$

$$\text{if } q = a, \text{ then } \quad (\overset{\frown}{\mathrm{a,b}} \cap \overset{\frown}{\mathrm{p,q}}) \in \{\emptyset, \overset{\frown}{\mathrm{p,b}}\}\,; \tag{3}$$

$$\text{if } \{p,q\} \subset \overset{\frown}{\mathrm{a,b}}, \text{ then } \quad \overset{\frown}{\mathrm{p,q}} \subset \overset{\frown}{\mathrm{a,b}} \quad \text{or} \quad \overset{\frown}{\mathrm{a,b}} \cap \overset{\frown}{\mathrm{p,q}} = \overset{\frown}{\mathrm{p,b}} \cup \overset{\frown}{\mathrm{a,q}}\,; \tag{4}$$

$$\text{if } \{a,b\} \subset \overset{\frown}{\mathrm{p,q}}, \text{ then } \quad \overset{\frown}{\mathrm{a,b}} \subset \overset{\frown}{\mathrm{p,q}} \quad \text{or} \quad \overset{\frown}{\mathrm{p,q}} \cap \overset{\frown}{\mathrm{a,b}} = \overset{\frown}{\mathrm{q,a}} \cup \overset{\frown}{\mathrm{b,p}}\,; \tag{5}$$

$$\text{if } p \in \overset{\frown}{\mathrm{a,b}} \text{ and } q \notin \overset{\frown}{\mathrm{a,b}}, \text{ then } \quad \overset{\frown}{\mathrm{a,b}} \cap \overset{\frown}{\mathrm{p,q}} = \overset{\frown}{\mathrm{p,b}}\,; \tag{6}$$

and

$$\text{if } p \notin \overset{\frown}{\mathrm{a,b}} \text{ and } q \in \overset{\frown}{\mathrm{a,b}}, \text{ then } \quad \overset{\frown}{\mathrm{a,b}} \cap \overset{\frown}{\mathrm{p,q}} = \overset{\frown}{\mathrm{a,q}}\,. \tag{7}$$

Proof. [(1)] We presume that $p = a$, the other case being susceptible to analogous treatment. If $q \in \overset{\frown}{\mathrm{a,b}}$, it follows from (7.25.2) that

$$\overset{\frown}{\mathrm{a,b}} = \overset{\frown}{\mathrm{a,q}} \cup \{q\} \cup \overset{\frown}{\mathrm{q,b}} \quad \Longrightarrow \quad \overset{\frown}{\mathrm{p,q}} = \overset{\frown}{\mathrm{a,q}} \subset \overset{\frown}{\mathrm{a,b}}\,. \tag{8}$$

Suppose, on the other hand that $q \notin \overset{\frown}{\mathrm{a,b}}$. If $q = b$, then $\overset{\frown}{\mathrm{p,q}} = \overset{\frown}{\mathrm{a,b}}$ so by (7.25.1) we may presume that q is in $\overset{\frown}{\mathrm{b,a}}$. From (7.25.2) follows that

$$\overset{\frown}{\mathrm{q,p}} \subset \overset{\frown}{\mathrm{b,a}} \quad \xrightarrow{\text{by (7.25.1)}} \quad \overset{\frown}{\mathrm{a,b}} \subset \overset{\frown}{\mathrm{p,q}}\,. \tag{9}$$

From (8) and (9) follows (1).

[(2)] Here we presume that $p = b$. Suppose that $q \in \overset{\frown}{\mathrm{a,b}}$. From (7.25.2) follows that

$$a \notin \overset{\frown}{\mathrm{q,b}} = \overset{\frown}{\mathrm{q,p}} \quad \xrightarrow{\text{by (7.25.1)}} \quad a \in \overset{\frown}{\mathrm{p,q}} \quad \xrightarrow{\text{by (7.25.2)}} \quad \overset{\frown}{\mathrm{p,a}} \subset \overset{\frown}{\mathrm{p,q}}\,. \tag{10}$$

Now suppose that $q \notin \overset{\frown}{\mathrm{a,b}}$. If $q = a$, then $\overset{\frown}{\mathrm{a,b}} \cap \overset{\frown}{\mathrm{p,q}} = \emptyset$ by (7.25.1) so we shall presume that $q \in \overset{\frown}{\mathrm{b,a}}$. From (7.25.2) follows that

$$\overparen{p,q} = \overparen{b,q} \subset \overparen{b,a} \xRightarrow{\text{by } (7.25.1)} \overparen{a,b} \cap \overparen{p,q} = \emptyset\,. \tag{11}$$

From (10) and (11) follows (2).

[(3)] The proof of (3) is analogous to that of (2).

[(4)] We presume that $\{p,q\} \subset \overparen{a,b}$. From (7.25.2) follows that $\overparen{a,p} \cup \{p\} \cup \overparen{p,b} = \overparen{a,b}$. If q is in $\overparen{p,b}$, then $\overparen{p,q} \cup \{q\} \cup \overparen{q,b} = \overparen{p,b}$ and so $\overparen{p,q} \subset \overparen{a,b}$. Thus we may presume that $q \in \overparen{a,p}$. From (7.25.2) follows that $\overparen{a,q} \cup \{q\} \cup \overparen{q,p} = \overparen{a,p}$ and so $\{\overparen{a,q}, \{q\}, \overparen{q,p}, \{p\}, \overparen{p,b}\}$ is a partition of $\overparen{a,b}$. Since (7.25.1) implies that $\overparen{p,q} \cap \overparen{q,p} = \emptyset$, we have $\overparen{a,b} \cap \overparen{p,q} = \overparen{p,b} \cap \overparen{a,q}$, which establishes (4).

[(5)] The proof of (5) is analogous to that of (4).

[(6)] We presume that $p \in \overparen{a,b}$ and $q \notin \overparen{a,b}$. From (7.25.1) follows that $\{\{a\}, \overparen{a,b}, \{b\}, \overparen{b,a}\}$ is a partition of M. From (7.25.2) follows that $\{\overparen{a,p}, \{p\}, \overparen{p,b}\}$ is a partition of $\overparen{a,b}$ and that $\{\overparen{b,q}, \{q\}, \overparen{q,a}\}$ is a partition of $\overparen{b,a}$. From these three facts follows that

$$\{\{a\}, \overparen{a,p}, \{p\}, \overparen{p,b}, \{b\}, \overparen{b,q}, \{q\}, \overparen{q,a}\} \text{ is a partition of } M\,. \tag{12}$$

Evidently (12) implies (6).

[(7)] The proof of (7) is analogous to that of (6). Q.E.D.

(7.27) Definition Let M be an exponential meridian with an orientation $\circlearrowright$. It follows from (7.26) that the family of all open arcs is a base for a topology, which we shall call the **arc topology**.[32] A necessary and sufficient condition for a subset of an exponential meridian to be open relative to the arc topology is for it to be a union of arcs. We recall that a topological space is **compact** if every open covering has a finite sub-covering.

(7.28) Theorem Let M be an exponential meridian which is compact relative to the arc topology. Then M is isomorphic to a circle meridian. In particular, relative to the arc topology, M is homeomorphic to a circle. Furthermore, if b is any basis for M, then F_b is isomorphic to the field of real numbers.

Proof. Let $\circlearrowright$ be the orientation of M of which $b = [0,1,\infty]$ is an element. On the field F_b we define the relation $\prec \subset (F_b \times F_b)$ by

$$(\forall\, \{x,y\} \subset F_b\,) \quad x \prec y \iff y - x \in \overbracket{0,\infty}\,. \tag{1}$$

For $x \in F_b$ we have

$$x - x = 0 \implies x - x \in \overbracket{0,\infty}\,. \tag{2}$$

For $\{x,y\} \subset F_b$, it follows from (7.9) that

$$\text{either } y - x \text{ or } x - y \text{ is in } M_{[0,1,\infty]}\text{, but not both.} \tag{3}$$

If $\{x,y,z\} \subset F_b$ and both $x \prec y$ and $y \prec z$, then

$$\{y-x, z-y\} \subset M_{[0,1,\infty]} \xRightarrow{\text{by } (7.14)} ((y-x)+(z-y) \in M_{[0,1,\infty]} \implies x \prec z\,. \tag{4}$$

It follows from (2), (3) and (4) that $\prec$ is a total order in the sense of (9.4). For $\{x,y,z\} \in F_b$ such that $x \prec y$, we have

$$y - x \in M_{[0,1,\infty]} \quad \text{and} \quad (z+y)\text{-}(z+x) = y - x \implies (z+x) \prec (z+y) \tag{5}$$

[32] We here adopt the convention that the empty set $\emptyset$ is an arc.

and for $\{x,y\} \subset M_{[0,1,\infty]}$ we have by (7.14)

$$xy \in M_{[0,1,\infty]} \implies 0 \prec xy \,. \tag{6}$$

It is a classical result of real analysis that if a field with a total order satisfies (2) through (6), then it is isomorphic with the field of real numbers if, and only if, to each subset of F_b with an upper bound corresponds a least upper bound. Let then $S \subset F_b$ and $b \in F_b$ be such that

$$(\forall\, x \in S) \quad x \prec b \,.$$

For $\{x,y\} \subset S$, we have that $x \prec b$ and $y \prec b$, whence follows that S is a directed set. Thus the identity function ι_S is a net and, since M is compact, there is a subnet of ι_S which converges to some element m of M. Specifically, this means that there exists a directed set D with direction $\acute{\prec}$ and a co-final function $\omega \big| D \hookrightarrow S$ such that $\iota_S \circ \omega$ converges to m.

Assume that m were not an upper bound for S. Then there would exist $s \in S$ such that $m \prec s$ and $m \neq s$. In that case there would exist $d \in D$ such that, for all $e \in D$ with $d \acute{\prec} e$,

$$s \prec \omega(e) \implies \iota_S \circ \omega \text{ is eventually outside of } \overparen{2m-s,s}$$

which, since m is in $\overparen{2m-s,s}$, would contradict the fact that $\iota_S \circ \omega$ converges to m. It follows that m is an upper bound of S.

Let u be any upper bound of S such that $u \prec m$. If $m \neq u$, then m would be in $\overparen{u,2m}$ and so $\iota_S \circ \omega$ would be eventually in $\overparen{u,2m}$. But, u being an upper bound of S, this would be absurd. It follows that m is a least upper bound for S. Q.E.D.

8. Appendix I: Mathematical Notation and Terminology

The language of modern mathematics is set theory. We set down here the many of the basic notions used in this paper and its sequel.

(8.1) Logical Notation We shall write

$$\exists \tag{1}$$

for the **existential quantifier** which is read as "there exists" or "for some": the notation

$$\exists! \tag{2}$$

is read as "there exists a unique" or "for exactly one". We write

$$\forall \tag{3}$$

for the **universal quantifier** which is read as "for all" or "for each" or "for every".
The notation

$$\Longrightarrow \tag{4}$$

is read "implies", the notation

$$\Longleftarrow \tag{5}$$

is read "follows from" and the notation

$$\Longleftrightarrow \tag{6}$$

is read "is a statement equivalent to the statement".

(8.2) Sets A **set** is a collection of objects. Such objects are called **members** or **elements** of the set. We denote that an object x is a member of a set X by

$$x \in X \quad \text{or} \quad X \ni x\,. \tag{1}$$

We denote than an object x is not a member of X by

$$x \notin X\,. \tag{2}$$

If X and Y are sets, then X is said to be a **subset of** Y unless

$$(\exists\, x \in X) \quad x \notin Y\,. \tag{3}$$

If X is a subset of Y, we say that Y is a **superset of** X: we express this in symbols by

$$X \subset Y \quad \text{or} \quad Y \supset X\,. \tag{4}$$

We convene that there is exactly one set with no elements, which we call **void**, or the **empty set** or the **null set**:

$$\emptyset\,. \tag{5}$$

For any set X, it follows from (3) that

$$\emptyset \subset X\,. \tag{6}$$

If x,...,y is a list of the elements of a set X, we express this fact by

$$X = \{x,\ldots,y\}\,. \tag{7}$$

A set with a single element is called a **singleton**, one with exactly two elements a **doubleton** *et cetera*.

If $P(x)$ is a condition on an object x, we denote the subset of a set X consisting of all those elements of X which satisfy the condition $P(x)$ by

$$\{x\in X : P(x)\}\,. \tag{8}$$

The symbol

$$\equiv \tag{9}$$

is used to define a symbol placed to the left of "≡", as the set which is placed to the left. For instance the **union of two subsets** X **and** Y of a third set Z is denoted and defined by

$$X\cup Y \equiv \{t\in Z : t\in X \text{ or } t\in Y\}\,. \tag{10}$$

The **intersection of two subsets** X and Y of a third set Z is denoted and defined by

$$X\cap Y \equiv \{t\in Z : t\in X \text{ and } t\in Y\}\,. \tag{11}$$

The terms **family** and **collection** are synonyms for the term set, although family is often reserved for sets, the elements of which are sets themselves. The family of all subsets of a given set X is denoted

$$\mathbf{2}^X\,. \tag{12}$$

Thus

$$S\in\mathbf{2}^X \Longleftrightarrow S\subset X\,. \tag{13}$$

For subsets X and Y of a set Z, the **symmetric difference of** X **and** Y is defined and denoted by

$$X\triangle Y \equiv \{t\in X\cup Y : t\notin X\cap Y\}\,. \tag{14}$$

When $Y\subset X$, the symmetric difference $X\triangle Y$ is often called the **complement of** Y **in** X.

The two **distributive laws** hold for any three subsets A, B and C of a set X:

$$\begin{aligned} A\cap(B\cup C) &= (A\cap B)\cup(A\cap C)\,;\\ A\cup(B\cap C) &= (A\cup B)\cap(A\cup C)\,. \end{aligned} \tag{15}$$

A third law, which is symmetric in ∪ and ∩ is

$$(A\cup B)\cap(B\cup C)\cap(C\cup A) = (A\cap B)\cup(B\cap C)\cup(C\cap A)\,. \tag{16}$$

The **De Morgan Laws** for subsets A and B of a set X are

$$\begin{aligned} X\triangle(A\cup B) &= (X\triangle A)\cap(X\triangle B)\,;\\ X\triangle(A\cap B) &= (X\triangle A)\cup(X\triangle B)\,. \end{aligned} \tag{17}$$

For any subfamily $\mathcal{S}\subset\mathcal{P}(X)$ of subsets of a set X, we denote

$$\bigcup_{S\in\mathcal{S}} S \equiv \{x\in X : (\exists\, S\in\mathcal{S})\ x\in S\} \quad\text{and}\quad \bigcap_{S\in\mathcal{S}} S \equiv \{x\in X : (\forall\, S\in\mathcal{S})\ x\in S\}\,. \tag{18}$$

(8.3) Cartesian Products A **pair** is a set of the form

$$\{x,y\} \tag{1}$$

where $x \neq y$. An **ordered pair of elements** x **and** y is defined and denoted by

$$[x,y] \equiv \{x,\{x,y\}\}. \tag{2}$$

We say that x is the **first coordinate** of the ordered pair [x,] and y the **second coordinate**. For subsets X and Y of a set Z, the **Cartesian product of** X **with** Y is defined and denoted by

$$X\times Y \equiv \{[x,y] : x\in X \text{ and } y\in Y\}. \tag{3}$$

More generally, a **triple** is a set {x,y,z} with exactly 3 elements, and an **ordered triple** is defined and denoted by

$$[x,y,z] \equiv \{x,\{x,y\},\{x,y,z\}\}. \tag{4}$$

Quadruples, **ordered quadruples**, **quintuples**, **ordered quintuples** *et cetera* are defined analogously.

(8.4) Finite Sets A subset S of a set X is a **proper subset** if $S \notin \{\emptyset, X\}$. A set X is said to be **infinite** if there exists a proper subset S of X and a subset $\prec$ of X×S such that

$$(\forall\, x\in X)(\exists!\, s\in S) \quad [x,s]\in\prec. \tag{1}$$

A set which is not infinite is called **finite**. Every subset of a finite set is finite.

(8.5) Conditions And Classes A **condition on a set** is a statement about a set which can be determined to either be **true** or **false**, but not both. A **class** is the aggregate of those sets for which some property is true. A set of a class is called a **member** of that class.

A family is always a class, but a class may not be a family. We demonstrate this to be true by what is known as "Russell's paradox", published by Bertrand Russell in 1901, but first enunciated by Ernst Zermelo in 1900:

Assume that all classes are families. Let $\mathcal{F}$ be the family of all infinite sets. Since $\mathcal{F}$ is infinite itself, it follows that $\mathcal{F}\in\mathcal{F}$. Let $\mathcal{C}$ be the family of all sets which are not elements of themselves. It is trivial that $\emptyset$ is an element of $\mathcal{C}$ and we have seen that $\mathcal{F}\notin\mathcal{C}$. If $\mathcal{C}\in\mathcal{C}$, then $\mathcal{C}$ is not in $\mathcal{C}$ by definition of $\mathcal{C}$. But if $\mathcal{C}\notin\mathcal{C}$, then $\mathcal{C}$ is in $\mathcal{C}$, again by the definition of $\mathcal{C}$. Thus our assumption cannot have been correct.

So how can we know if a class is a family (set)? If a class has been constructed from sets using unions, intersections, subsets or Cartesian products, we can be sure it is a set. Ernst Zermelo suggested another method, called the **axiom of choice**: to wit

$$(\forall\, \mathcal{S} \text{ a family of sets})(\exists \text{ a set } X)(\forall\, S\in\mathcal{S})(\exists\, x_S\in S) \quad X\cup S = \{x_S\}. \tag{1}$$

Such a function

$$\mathcal{S}\ni S \hookrightarrow x_S\in S$$

is called a **choice function**. It is known that if one constructs new sets using the axiom of choice, one is not thereby led into logical contradictions. However there is no way to deduce that the axiom of choice holds from the other axioms of set theory. One can use it or not, depending on ones taste.

(8.6) Equivalence Relations The terms "relation" and "graph" are in some places treated as synonyms. Here we shall treat the term graph as being more restrictive than a relation, and shall reserve the next section for its presentation.

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes or sets. By a **relation from** $\mathcal{A}$ to $\mathcal{B}$, we shall mean a class of ordered pairs $[\mathrm{A,B}]$ where each A is a member of $\mathcal{A}$ and each B a member of $\mathcal{B}$.

if $\mathcal{A} = \mathcal{B}$ and the following three conditions hold, we say that a relation $\sim$ from $\mathcal{A}$ to $\mathcal{B}$ is an **equivalence relation**:

$$(\forall\, \mathrm{A} \in \mathcal{A}) \quad [\mathrm{A,A}] \in \sim\,; \tag{1}$$

$$(\forall\, \{\mathrm{A,B}\} \subset \mathcal{A}) \quad [\mathrm{A,B}] \in \sim\; \Longleftrightarrow [\mathrm{B,A}] \in \sim\,; \tag{2}$$

$$(\forall\, \{\mathrm{A,B,C}\} \subset \mathcal{A}\colon [\mathrm{A,B}] \in \sim \text{ and } [\mathrm{B,C}] \in \sim) \quad [\mathrm{A,C}] \in \sim\,. \tag{3}$$

An **equivalence class associated with the equivalence relation** $\sim$ **and a member** A **of** $\mathcal{A}$ is

$$\text{the class of all members B of } \mathcal{A} \text{ such that } [\mathrm{A,B}] \text{ is a member of } \mathcal{A}\,. \tag{4}$$

The **partition of** $\mathcal{A}$ **associated with the equivalence relation** $\sim$ is the class of all equivalence classes associated with $\sim$.

Conversely, for a class or set $\mathcal{A}$ we define a **partition of** $\mathcal{A}$ to be a class $\mathcal{P}$ of sub-classes of $\mathcal{A}$ such the such that each member of $\mathcal{A}$ is a member of a unique member of $\mathcal{P}$. The **equivalence relation** $\sim$**associated with the partition** $\mathcal{P}$ is the class of all ordered pairs $[\mathrm{X,Y}]$ where X and Y are members of a common member of $\mathcal{P}$.

9. Appendix II: Graphs

Many things can be described in the context of a subset of the Cartesian product of two sets. We present some of the more important ones in the present section.

(9.1) Definitions A subset $\prec$ of a Cartesian product $X\times Y$ of sets is said to be a **graph of X with Y**. For $x\in X$ and $y\in Y$, we shall sometimes write

$$x \prec y \tag{1}$$

to indicate that $[x,y]$ is an element of $\prec$ and

$$x \not\prec y\,, \tag{2}$$

to indicate that $[x,y]$ is not an element of $\prec$. To indicate that $[x,y]\subset\prec$ but $x\neq y$, we shall write

$$x \overset{!}{\prec} y\,. \tag{3}$$

The **domain of a graph** $\prec\subset X\times Y$ is defined and denoted by

$$\lfloor\preceq\rfloor \equiv \{x\in X : (\exists\, y\in Y)\ [x,y]\in\prec\} \tag{4}$$

and its **range** by

$$\lceil\succeq\rceil \equiv \{y\in Y : (\exists\, x\in X)\ [x,y]\in\prec\}\,. \tag{5}$$

The **world of a graph** $\prec$ is defined and denoted by

$$\boxed{\prec} \equiv \lfloor\preceq\rfloor\cup\lceil\succeq\rceil\,. \tag{6}$$

The **inverse of a graph** $\prec$ is defined and denoted by

$$\prec^{-1} \equiv \{[y,x] : [x,y]\in\prec\}\,. \tag{7}$$

In particular

$$(X\times Y)^{-1} \;=\; Y\times X\,. \tag{8}$$

The **complement of a graph** $\prec$ is the set

$$\triangle\prec \;\equiv\; \{[x,y]\in\lfloor\preceq\rfloor\times\lceil\succeq\rceil : [x,y]\not\in\prec\}\,. \tag{9}$$

If $\prec$ and σ are graphs, then the **composition of $\prec$ with σ** is defined and denoted by

$$\prec\circ\sigma \equiv \{[x,z] : (\exists\, y\in\lfloor\sigma\rfloor)\ [x,y]\in\prec \text{ and } [y,z]\in\sigma\}\,. \tag{10}$$

(9.2) Functions A graph ϕ is a **function** if

$$(\forall\, x\in\lfloor\phi\rfloor)(\exists!\, y\in\lceil\phi\rceil)\ [x,y]\in\phi\,. \tag{1}$$

For a function ϕ and $[x,y]\in\phi$, we define the notation

$$\phi(x) \equiv y\,. \tag{2}$$

A function $\phi\subset X\times Y$ is said to be **surjective** if $\lceil\phi\rceil=Y$ and $\lfloor\phi\rfloor=X$. A function is said to be **injective** if its inverse is also a function. A function is **bijective** if it is both injective and surjective.

If $\lceil\phi\rceil=\lfloor\phi\rfloor$ for a bijective function, we say that ϕ is a **permutation of its domain**. A subset S of the domain of a permutation ϕ such that $\phi(s)\in S$ for each $s\in S$ is said to be **invariant**. An invariant set containing no proper invariant subsets is said to be an **orbit of** ϕ. If ϕ and θ are permutations with a common domain, and if each orbit of θ is a subset of some orbit of ϕ, we shall say that θ is a **refinement of** ϕ.

The notation

$$\phi\big| X \hookrightarrow Y \tag{3}$$

means that ϕ is a function such that $\lfloor\phi\rfloor = X$ and $\ulcorner\phi\urcorner \subset Y$. This is sometimes described by saying that ϕ **is a function from** X **into** Y (when $\ulcorner\phi\urcorner = Y$ the term "into" can be replaced by "onto"). If $E(x)$ denotes an expression in terms of x, then

$$\phi\big| X \ni x \hookrightarrow E(x) \in Y \tag{4}$$

indicates in addition to (3) that

$$(\forall\, x\in X) \quad \phi(x) = E(x)\,. \tag{5}$$

A statement of the form

$$X \ni x \hookrightarrow E(x) \in Y \tag{6}$$

defines a function without giving it a name.

The **identity function on a set** X is the function

$$\iota_X \equiv \{[x,x] : x\in X\}\,. \tag{7}$$

For any bijective function $\phi\subset X\times Y$, we have

$$\phi^{-1}\circ\phi = \iota_X \quad \text{and} \quad \phi\circ\phi^{-1} = \iota_Y\,. \tag{8}$$

A function of which the range is a singleton is called a **constant function**: if $X = \lfloor\phi\rfloor$ and $\{c\} = \ulcorner\phi\urcorner$, we write

$$\overset{c}{\underset{X}{\rightarrow}} \quad \text{for the function} \quad X\ni x\hookrightarrow c\,. \tag{9}$$

Associated with any graph $\prec\,\subset X\times Y$ are four "set functions":

$$\overrightarrow{\prec}\big|\mathcal{P}(X)\ni S\hookrightarrow\{y\in Y : (\exists\, x\in S)\ [x,y]\in\prec\}\in\mathcal{P}(Y)\,, \tag{10}$$

$$\overleftarrow{\prec}\big|\mathcal{P}(Y)\ni S\hookrightarrow\{x\in X : (\exists\, y\in S)\ [x,y]\in\prec\}\in\mathcal{P}(X)\,, \tag{11}$$

$$\overset{\frown\circ}{\prec}\big|\mathcal{P}(X)\ni S\hookrightarrow\{y\in Y : (\forall\, x\in S)\ [x,y]\in\prec\}\in\mathcal{P}(Y) \tag{12}$$

and

$$\overset{\circ\frown}{\prec}\big|\mathcal{P}(Y)\ni S\hookrightarrow\{x\in X : (\forall\, y\in S)\ [x,y]\in\prec\}\in\mathcal{P}(X)\,. \tag{13}$$

We have in general

$$\overrightarrow{\prec} = \overleftarrow{\prec^{-1}}\,, \quad \overleftarrow{\prec} = \overrightarrow{\prec^{-1}}\,, \quad \overset{\frown\circ}{\prec} = \overset{\circ\frown}{\prec^{-1}} \quad \text{and} \quad \overset{\circ\frown}{\prec} = \overset{\frown\circ}{\prec^{-1}}\,. \tag{14}$$

These two set functions are actually determined by their values at the singletons: to exploit this fact, we define

$$\overrightarrow{\prec}\big|X\ni x\hookrightarrow\overrightarrow{\prec}(\mathbf{x})=\overrightarrow{\prec}^{\circ}(\mathbf{x})=\{y\in Y:[x,y]\in\prec\}\in\mathcal{P}(Y) \tag{15}$$

and

$$\overleftarrow{\prec}\big|Y\ni y\hookrightarrow\overleftarrow{\prec}(\mathbf{y})={}^{\circ}\overleftarrow{\prec}(\mathbf{y})=\{x\in X:[x,y]\in\prec\}\in\mathcal{P}(X)\,. \tag{16}$$

We have

$$(\forall\, S\subset X)\quad \overrightarrow{\prec}(S)=\bigcup_{x\in S}\overrightarrow{\prec}(x)\,,\quad (\forall\, S\subset Y)\quad \overleftarrow{\prec}(S)=\bigcup_{y\in S}\overleftarrow{\prec}(y)\,, \tag{17}$$

$$(\forall\, S\subset X)\quad \overrightarrow{\prec}^{\circ}(S)=\bigcap_{x\in S}\overrightarrow{\prec}(x)\quad\text{and}\quad(\forall\, S\subset Y)\quad {}^{\circ}\overleftarrow{\prec}(S)=\bigcap_{y\in S}\overleftarrow{\prec}(y)\,. \tag{18}$$

If S is any subset of the domain of an order $\prec$, then the **restriction of $\prec$** to S is defined and denoted by

$$\prec\big|_S\,. \tag{19}$$

For sets X and Y, the set of all functions from X into Y is sometimes denoted by

$$X^Y \tag{20}$$

We shall extend this notation to

$$Y^{X!}\equiv\{\phi\in Y^X:\phi\text{ is bijective}\}\,. \tag{21}$$

Two functions $\phi\big|X\hookrightarrow Y$ and $\theta\big|X\hookrightarrow Y$ will be said to be **compatible** if

$$(\forall\,\{x,y\}\subset X)\quad \phi(x)=\phi(y)\Longleftrightarrow\theta(x)=\theta(y)\,. \tag{22}$$

Compatibility of functions ϕ and θ can be expressed more succinctly as

$$\phi\circ\theta^{-1}\text{ is a function.} \tag{23}$$

A graph $\prec$ will be said to be **symmetric** provided that there exists a bijection $\phi\big|\llcorner\!\prec\!\lrcorner\hookrightarrow\ulcorner\!\prec\!\urcorner$ such that

$$(\forall\,[a,b]\in\llcorner\!\prec\!\lrcorner\times\ulcorner\!\prec\!\urcorner)\quad [a,b]\in\prec\quad\Longleftrightarrow\quad[\phi^{-1}(b),\phi(a)]\in\prec\,. \tag{24}$$

(9.3) Equivalence The concept of an equivalence relation was introduced in (8.6). When the symbol $\mathcal{A}$ there represents a set, the equivalence relation $\sim$ is a graph. Such an equivalence relation is distinguished by the following:

$$\iota_{\mathcal{A}}\subset\sim,\quad \sim=\sim^{-1}\quad\text{and}\quad(\sim\circ\sim)\subset\sim\,. \tag{1}$$

The three parts of (1) may also be expressed by

$$(\forall\, x\in\mathcal{A})\ x\sim x\,,\quad(\forall\,\{x,y\}\subset\mathcal{A})\ x\sim y\Longleftrightarrow y\sim x\quad\text{and}\quad(\forall\,\{x,y,z\}\subset\mathcal{A}:\ x\sim y\text{ and }y\sim z)\ \ x\sim z\,.$$

(9.4) Order An **order on a set** X is a relation $\prec\subset X\times X$ such that

$$\iota_X=\prec\cap\prec^{-1}\quad\text{and}\quad(\prec\circ\prec)\subset\prec\,. \tag{1}$$

Another way of expression the two conditions of (1) is

$$(\forall\, x\in X)\ x\prec x\,,\quad(\forall\,\{x,y\}\subset X:\ x\prec y\text{ and }y\prec x)\ x=y\quad\text{and}\quad(\forall\,\{x,y,z\}\subset X:\ x\prec y\text{ and }y\prec z)\ x\prec z\,.$$

An order is sometimes called a **partial ordering** (or simply an **ordering**) and, relative to $\prec$, X is called an **ordered set**, or **partially ordered set** or **poset**.

An order $\prec$ is said to be **linear** or **total** if

$$X\times X = (\prec\cup\prec^{-1}) \tag{2}$$

or, what amounts to the same thing, if

$$(\forall\,[x,y]\in X\times X) \quad \text{either} \quad x\prec y \text{ or } y\prec x\,. \tag{3}$$

A subset σ of an order $\prec$ such that $\llcorner\sigma\lrcorner\times\llcorner\sigma\lrcorner = \sigma\cup\sigma^{-1}$ which is called a **chain** in $\prec$. The **Hausdorff Maximality Principle**, which is equivalent to the Axiom of Choice, asserts that each chain is a subset of some maximal chain.

The **supremum** of a subset S of $\llcorner\prec\lrcorner$, if it exists, is an element $\mathfrak{sup}(S)$ such that

$$(\forall\, s\in S)\ s\prec\mathfrak{sup}(S) \quad \text{and} \quad (\forall\, x\in S\colon (\forall\, s\in S)\ s\prec x) \quad \mathfrak{sup}(S)\prec x\,. \tag{4}$$

The **infimum** of a subset S of $\llcorner\prec\lrcorner$, if it exists, is an element $\mathfrak{inf}(S)$ such that

$$(\forall\, s\in S)\ \mathfrak{inf}(S)\prec s \quad \text{and} \quad (\forall\, x\in S\colon (\forall\, s\in S)\ x\prec s) \quad x\prec\mathfrak{inf}(S)\,. \tag{5}$$

For x and y in an ordered set, we define (if they exist)

$$x\wedge y \equiv \mathfrak{inf}(\{x,y\}) \quad \text{and} \quad x\vee y \equiv \mathfrak{sup}(\{x,y\})\,. \tag{6}$$

(9.5) Cardinality If ϕ is any bijective function, then $\llcorner\phi\lrcorner$ and $\ulcorner\phi\urcorner$ are said **to have the same cardinality**. Because each identity function is bijective, each set has the same cardinality as itself. Furthermore, if sets X and Y have the same cardinality and Y and Z have the same cardinality, then X and Z have the same cardinality. Thus it is possible, for any set X, to attach a symbol to it, and by extension, attach this same symbol to every other set with the same cardinality as X. There are many equivalent ways to do this for common sets. Here are some typical examples:

We attach the symbol 0 to the empty set $\varnothing$ and the symbol 1 to the singleton $\{\varnothing\}$. Attach the symbol 2 to the ordered pair $[1,\varnothing]$, 3 to the ordered pair $[2,\varnothing]$, 4 to the the ordered pair $[3,\varnothing]$ and so on *ad infinitum.* The set

$$\mathbb{N} \tag{1}$$

of all such symbols distinct from 0 is called the set of **natural numbers**. We attach the symbol $\aleph_0$ to the set $\mathbb{N}$. Any set which (by extension) has one of these symbols attached to it is said to be **countable**. If such symbol is attached to some set, this symbol is said to be its **cardinality**. These symbols are called **cardinal numbers**. If the cardinality of a non-void set is an element of $\mathbb{N}$, it is said to be **finite**. This definition of "finite" is consistent with the one given in (8.4).

A set of which the cardinality is $\aleph_0$ is said to be **countably infinite**. We denote the cardinality of any set X by

$$\#X \tag{2}$$

and, for any natural number n, we denote by

$$\underline{n} \tag{3}$$

the set of natural numbers less than or equal to n. Any function with domain of the form $\underline{n}$ is said to be a **finite sequence**. An **infinite sequence** is a function with domain $\mathbb{N}$. For a sequence σ and $m\in\llcorner\sigma\lrcorner$, the m-th **element of** σ is defined and denoted by

$$\sigma_m \equiv \sigma(\mathrm{m}) . \tag{4}$$

For any $\mathrm{n} \in \mathbb{N}$ and any set X, the elements of

$$\mathrm{X}^{\underline{\mathrm{n}}} \tag{5}$$

are called **n-tuples**: the sequences from $\underline{\mathrm{n}}$ into X. Such an n-tuple x is frequently written

$$[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}] . \tag{6}$$

Beginning with 1, every other finite cardinal number is said to be **odd**. The other finite cardinal numbers are **even**. A set is said to be **even** if its cardinality is even – **odd** if its cardinality is odd. Two finite sets **have equal parity** if they are both even, or both odd.

10. Appendix III: Topology

(10.1) Definitions A **topology** is a family $\mathcal{T}$ of subsets of a set X containing X, the empty set $\emptyset$, the union of the members of any subfamily of $\mathcal{T}$ and the intersection of the members of any finite subfamily of $\mathcal{T}$ .The members of a topology are said to be **open** subsets of X and complements of open sets are said to be **closed** subsets of X. A subset K of X is said to be **compact** if, whenever K is a subset of the union of a subfamily of $\mathcal{T}$, then it is also subset of some finite subfamily of that "**covering**" subfamily. A subset of X is **disconnected** if it is the union of two disjoint elements of $\mathcal{T}$, and **connected** if it is not disconnected. A subset N of X is said to be a **neighborhood** of one of its elements x if there exists $O\in\mathcal{T}$ such that $x\in O$ and $O\subset N$. A topology is said to be a **hausdorff** topology if each, for all distinct x and y in X, there exist disjoint open sets U and V such that $x\in U$ and $y\in V$.

(10.2) Relative Topology If $\mathcal{T}$ is a topology on a set X and if S is a subset of X, then the family $\{O\cap S : O\in\mathcal{T}\}$ is a topology on S called the **relativized topology**. The set S is compact for its relativized topology if, and only if, it is compact as a subset of X. The same is true for the concepts "connected" and "hausdorff".

(10.3) Base for a Topology Let X be a set and $\mathcal{B}$ a family of subsets of X such that the intersection of any two elements of $\mathcal{B}$ is an element of $\mathcal{B}$. Suppose that the null set is in $\mathcal{B}$ and that each element of x is in some element of $\mathcal{B}$. Let $\mathcal{T}$ be the family of all subsets $S\subset X$ such that S is a union of elements of $\mathcal{B}$. Then $\mathcal{T}$ is a topology for X. We say that $\mathcal{B}$ is a **base for the topology** $\mathcal{T}$.

(10.4) Nets Let $\mathcal{T}$ be a topology on a set X. An order $\prec$ on a set D is said to be a **direction on** D provided

$$(\forall \{x,y\}\subset D)(\exists z\in D) \quad x\prec z \text{ and } y\prec z\,. \tag{1}$$

Any function with domain a directed set is said to be a **net**. A net ω from one directed set D with direction $\prec$ into another directed set E with direction $\acute{\prec}$ is said to be **co-final** if

$$(\forall e\in E)(\exists d\in D)(\forall x\in D:\ d\prec x) \quad e\,\acute{\prec}\,\omega(x)\,. \tag{2}$$

If D and E are as above, ω is a co-final net in E and if ν is a net with domain E, then $\nu\circ\omega$ is said to be a **subnet** of ν. A net defined on a directed set D is said to be **eventually in a set** S if

$$(\exists d\in D)(\forall x\in D:\ d\prec x) \quad \nu(x)\in S \tag{3}$$

and to be **frequently in a set** S if

$$(\forall d\in D)(\exists x\in D) \quad d\prec x \quad \text{and} \quad \nu(x)\in S\,. \tag{4}$$

A net ν is said to **converge** to an element x of X if

$$(\forall O\in\mathcal{T}:\ x\in O) \quad \nu \text{ is eventually in } O\,. \tag{5}$$

The point x is said to be the **limit** of the net ν.

The proofs of the following facts are pedestrian:

$$(\forall S\subset X) \quad S \text{ is open} \iff \text{each net convergent to a point in S is eventually in S}, \tag{6}$$

$$(\forall S\subset X) \quad S \text{ is closed} \iff \text{the limit of any convergent net in S is in S}, \tag{7}$$

$$(\forall\, S \subset X) \quad S \text{ is compact} \iff \text{each net in } S \text{ has a subnet convergent to an element of } S. \tag{8}$$

(10.5) Continuity Let $\mathcal{T}_1$ be a topology on a set X_1 and $\mathcal{T}_2$ be a topology on a set X_2. A function $f\big|\ X_1 \hookrightarrow X_2$ is said to be **continuous** provided

$$(\forall\, O \in \mathcal{T}_2) \quad \overleftarrow{f}(O) \in \mathcal{T}_1 \tag{1}$$

or, equivalently

$$(\forall\, \omega \text{ convergent to } w \in X_1) \quad f \circ \omega \text{ converges to } f(w) \text{ in } X_2\,. \tag{2}$$

The proofs of the following facts are pedestrian:

$$\text{the composition of any two continuous functions is continuous,} \tag{3}$$

$$(\forall\, S \subset X_1 : S \text{ is compact})(\forall\, f\big|\ X_1 \hookrightarrow X_2 \text{ continuous}) \quad \overrightarrow{f}(S) \text{ is compact in } X_2 \tag{4}$$

and

$$(\forall\, S \subset X_1 : S \text{ is connected})(\forall\, f\big|\ X_1 \hookrightarrow X_2 \text{ connected}) \quad \overrightarrow{f}(S) \text{ is connected in } X_2\,, \tag{5}$$

(10.6) Metrics A **metric** d is a non-negative function on the cartesian product of a set X with itself such that[33]

$$(\forall\, \{x,y\} \subset X) \quad d([x,y]) = 0 \iff x = y\,, \tag{1}$$

$$(\forall\, \{x,y\} \subset X) \quad d([x,y]) = d([y,x]) \tag{2}$$

and

$$(\forall\, \{x,y,z\} \subset X) \quad d([x,y]) + d([y,z]) \leq d([x,z])\,. \tag{3}$$

The **ball of radius** r **around a point** x in X, for $r \geq 0$, is the set

$$\{t \in X : d([x,t]) \leq r\}\,. \tag{4}$$

The family of all such balls forms a base for a **metric topology**.

A metric topology is Hausdorff. A subset of a metric space is compact if, and only if, each sequence has a convergent subsequence.

[33] The value $d([x,y])$ of d at an ordered pair $[x,y]$ is usually abbreviated to $d(x,y)$.

11. Notation

12. Index

13. Eponymy

Abelian (group and libra): Henrik Abel was born in Nedstrand, Norway in 1802 and died in 1829.

Euclidean (plane and space): Euclid of Alexandria lived in Greece and Egypt *circa* 300 BC.

Gaussian (plane): Johann Carl Friedrich Gauss was born in Braunschweig, Germany in 1777 and died in 1855.

Hausdorff (topological space and maximality principle): Felix Hausdorff was born in Breslau, Germany in 1868 and died in 1942.

Klein (4-group): Christian Felix Klein was born in Düsseldorf, Prussia in 1849 and died in 1925.

Riemann (sphere): Bernhard Riemann was born in Brelenz, Hanover in 1826 and died in 1866.

14. Acknowledgement

This work, in part, was prosecuted at the Division of Information Sciences at Academia Sinica in Taipei, Taiwan. The author wishes to thank that institution for providing facilities and specifically Professors Hsu Tsan-Sheng and Kao Ming-Tat for their help, friendship and hospitality.

15. Bibliography

HEWITT, EDWIN AND KARL STROMBERG: Real and Abstract Analysis. Springer Verlag 1969.

SEIDENBERG,ABRAHAM: Lectures in Projective Geometry. Princeton: D. Van Nostrand Company, Inc. 1962

TITS, JACQUES: Généralisation des groupes projectifs basés sur la notion de transitivité. Mem. Acad. Roy. Beg. **27** (1952), 115 p.

VEBLEN, OSWALD AND JOHN WESLEY YOUNG: Projective Geometry, Vol. I. Boston: Ginn and Company, 1910.

VON STAUDT, KARL: Geometrie der Lage. Nürnberg: Verlag der Friedr. Korn'schen Buchhandlung, 1847.