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# A Micro-Matching Foundation of Neutral Technical Progress

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Abstract: We develop a general two-sided micro-matching framework with heterogeneous workers and machines that permits a complete analysis of neutral technical progress commonly used in neoclassical production theory, without *a priori* restrictions on the functional forms of the production technology. We use the concept of “production core” to determine stable task assignments and the corresponding factor-return distributions, and then examine how these equilibrium outcomes respond to neutral technical progress pertaining to a particular worker or to all factors. Technical progress that is uniform in all factors will not alter equilibrium micro-matching. Technical progress of the labor-augmenting type may (i) cause a “turnover” by destroying existing stable task assignments and creating new stable task assignments, (ii) generate a factor-return redistribution similar to Harrod-neutral technical progress in neoclassical theory, and (iii) create “spillover” effects from the innovating worker to his/her potential matching machines and his/her directly and indirectly competing workers. The possibility of turnovers and the extent to which factor returns are redistributed depend on the value of the current matches, the extent of outside threats from latent technologies, and the size of technical progress.

JEL Classification: D20, C71, O33.

Keywords: micro-matching, stable assignment, neutral technical progress, turnover.

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# 1 Introduction

Rooted in the classic work by Walras (1874), Marshall (1890), Wicksteed (1894), and Wicksell (1900), neoclassical production theory has been the sole framework for studying the microeconomic behavior of a firm’s production activity and the aggregate performance of an economy. In a series of contributions by Allen (1938), Solow (1957), Arrow, Chenery, Minhas and Solow (1961) and Samuelson (1962), an array of well-behaved, often constant-elasticity-of-substitution (CES), neoclassical production functions have been constructed. Because these functions are easily understood and highly tractable, their applications to economic theory have created many valuable analytic results. Yet it is well-known that neoclassical production theory is rather restricted when there exist vast heterogeneities in multiple production factor inputs. On the one hand, such continuous functions impose, *a priori*, direct restrictions on factor income distributions.<sup>1</sup> On the other hand, it is essentially impossible for this approach to accommodate any higher dimension of heterogeneities than 2-by-2 in a tractable manner.<sup>2</sup> To permit flexibility in factor heterogeneities is important especially if one is interested in better understanding the consequences of factor-specific technical progress for factor-return redistributions where productivity depends on underlying heterogeneities.

In order to enable rich factor heterogeneities, we propose two-sided micro-matching theory as an organizing framework that avoids imposing, *a priori*, functional-form restrictions on factor income distributions. The mathematical foundation of our research is based upon the pivotal work by von Neumann (1953) and the techniques developed in our paper are applications of combinatorics and lattice theory. Specifically, the determination of micro-matching by each production unit is viewed as a linear assignment problem, whereas the equilibrium concept is “production core” that induces efficient outcomes. In this vein, the basic analysis is largely completed in the seminal pieces by Shapley and Shubik (1972), Crawford and Knoer (1981) and Roth and Sotomayer (1990).<sup>3</sup> Our paper generalizes this existing micro-matching literature by introducing an important element in modern production theory, namely, technical progress. We provide a complete analysis on how the patterns of micro-matching and the resulting factor-return distributions respond to technological advancements. This paper can therefore be viewed as moving micro-matching theory toward the

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<sup>1</sup>This is so even with general CES or non-homothetic CES functions (cf. Sato 1975 and Shimomura 1999).

<sup>2</sup>In the 2-by-1 case, Fallon and Layard (1976) propose a two-level CES production function to model two types of labor with homogeneous capital. In the 2-by-2 case, Krusell, Ohanian, Rios-Rull and Violante (2000) construct a three-level CES production function.

<sup>3</sup>Strictly speaking, this class of models features n-by-m disjoint agents with continuous payoffs.

contemporary literature of dynamics, thus enabling fruitful future applications to dynamic productivity analysis, growth theory, and real business cycle studies.<sup>4</sup>

More specifically, we consider a “microeconomy” with a finite number of heterogeneous workers and a finite number of heterogeneous machines. A firm’s production technology is described by two-sided micro-matching between workers and machines, without *a priori* restrictions on the functional forms.<sup>5</sup> The value of output can vary for any particular pair of worker and machine. Using the concept of “production core” *a la* von Neumann (1953), we determine “stable task assignments” that describe the pattern of micro-matching between workers and machines associated with manifest technologies. Those not in the production core represent latent technologies, which become “outside alternatives” to stable task assignments. Thus, any relative advancements in such technologies can change the nature of micro-matching between workers and machines, contrasting sharply with neoclassical production theory that accounts only for manifest technologies. The consideration of outside alternatives in constructing an equilibrium is a crucial feature in game-theoretic models, but is omitted in the Walrasian general equilibrium on which neoclassical production theory is based. Summarizing our contribution, the framework proposed in this paper not only grants flexibility in factor input heterogeneities, but permits both manifest and latent technologies to play important roles in determining equilibrium.

Upon constructing a framework of micro-matching with on-going technical progress, we proceed with a complete characterization of the distribution of factor returns. We then undertake a thorough examination of how various types of neutral technological advancements, particularly those commonly used in neoclassical theory, may influence stable task assignments within each production unit and the resulting redistribution of factor returns. We focus primarily on two widely used forms of neutral technical progress: one pertaining to all workers and machines and another exclusively to a particular worker regardless of the matching machine. While the former represents a basic form of disembodied technical progress that is uniform in all production factors, the latter is purely labor-augmenting. These types of neutral technical progress are of particular interest because their counterparts in neoclassical production theory are called Harrod-neutral and Hicks-neutral, which

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<sup>4</sup>The reader should be reminded that we are talking about micro-matching theory, not macro-matching based on aggregate (often random) matching functions which can easily incorporate growth dynamics as in Laing, Palivos and Wang (1995) and Chen, Mo and Wang (2002).

<sup>5</sup>Although we have restricted our attention to production theory, the two-sided matching structure can be easily applied to the college admissions game and the marriage game considered by Gale and Sharpley (1962).

are widely used in studies in economic dynamics (cf. Uzawa 1961 and Wan 1971).

Our main findings can be summarized below. First, technical progress that is uniform to all factors will not alter equilibrium micro-matching, while technical progress of the labor-augmenting type may cause a “turnover” by destroying existing stable task assignments and creating new stable task assignments. Second, whether technical progress of the labor-augmenting type leads to a turnover depends crucially on the value of the current matches, the extent of outside threats from latent technologies, and the size of technical progress. Third, under technical progress of the labor-augmenting type for a particular worker, the properties obtained in our micro-matching framework contain the neoclassical features, by including a factor-return redistribution similar to Harrod-neutral technical progress in neoclassical theory as an equilibrium outcome. Fourth, even with a neoclassical Harrod-neutral distribution, the innovating worker does acquire the entire microeconomy-wide gain, though such a gain may be greater or less than the direct incremental value of production created by the manifest technology associated with the innovating worker. Finally, technical progress of the labor-augmenting type for a particular worker can create “spillover effects” on factor returns to the innovating worker’s potential mates (machines) and his/her directly and indirectly competing workers. In particular, this type of technical progress causes disadvantages for the worker losing his/her machine to the innovating worker relative to the worker taking over an innovating worker’s old mate and others indirectly competing workers; it grants the innovating worker’s new mate advantages over the innovating worker’s old mate and other potential mates.

The remainder of the paper is organized as follows. Section 2 constructs a two-sided micro-matching framework with heterogeneous workers and machines, and defines stable assignments and stable factor-return distributions. Section 3 defines the equilibrium based on the concept of production core and the sets of equilibrium distributions associated with two types of neutral technical progress. In Section 4, we study how each type of neutral technical progress may influence stable assignments and equilibrium factor-return distributions. Finally, we summarize the main properties established and propose some avenues of future research in the concluding section.

## 2 The Basic Framework

We focus on characterizing two-sided micro-matching between workers and machines within each production unit, say, a firm. There are  $n \geq 2$  workers and  $m \geq 2$  machines. Denote the set of workers within the firm of our consideration as  $L$ , the set of machines as  $K$ , and the set of “agents”

as  $A = L \cup K$ . A **task**  $(i, j)$  consists of a pair of worker and machine  $(\ell_i, k_j)$  where  $\ell_i \in L$  and  $k_j \in K$ . Each task creates a payoff  $v_{ij} \geq 0$ . A **microeconomy**  $\mathcal{V}$  is represented by a payoff matrix  $(v_{ij})$  that summarizes all the payoffs associated with different tasks.<sup>6</sup>

An **assignment**, denoted by  $\mu$ , is a list of tasks with no worker or machine involving in more than one task:

$$\mu = \{(i, j) \mid \text{each } i \text{ and } j \text{ is matched at most once, for } i = 1, \dots, n \text{ and } j = 1, \dots, m\} \quad (1)$$

Thus, an assignment describes *potential* micro-matching between workers and machines. Denote the set of all possible assignments as  $[\mu]$ . The **value** of production associated with an assignment  $\mu$  is measured by,

$$V(\mu) = \sum_{(i,j) \in [\mu]} v_{ij} \quad (2)$$

Obviously, our production technology satisfies the neoclassical constant-returns-to-scale property, that is, increasing the numbers of workers and machines proportionately will lead to an increase in the value of production in the same scale.

**Definition 1:** An **efficient** assignment  $\mu^e \in [\mu]$  is an assignment such that  $V(\mu^e) \geq V(\mu)$  for all  $\mu \in [\mu]$ .

The most important step toward determining an equilibrium of the microeconomy is to specify the distribution of factor returns. Let  $w_i$  and  $z_j$  denote the returns to worker  $\ell_i \in L$  and to machine  $k_j \in K$ , respectively.

**Definition 2:** A **distribution of factor returns**  $X(\mu) = (w_1, \dots, w_n, z_1, \dots, z_m)$  is one such that  $w_i \geq 0$ ,  $z_j \geq 0$ , and  $w_i + z_j = v_{ij}$  for all  $(i, j) \in \mu$ .

Then, we consider,

**Definition 3:** A **stable** assignment is an efficient assignment  $\mu^* \in [\mu]$  associated with stable factor-return distributions  $X^*(\mu^*) = (w_1, \dots, w_n, z_1, \dots, z_m)$  such that

$$w_i + z_j \geq v_{ij} \quad \text{for all } (i, j) \in \mu \quad (3)$$

$$w_i + z_j = v_{ij} \quad \text{for } (i, j) \in \mu^*. \quad (4)$$

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<sup>6</sup>Thus, we implicitly assume that all agents have linear utility. However, the reader may see upon examining our paper that our main results can be easily extended to the case of nonlinear utility by applying Demange and Gale (1985).

The *set of stable assignments* is denoted as  $[\mu^*]$  and the *set of stable factor-return distributions* is denoted as  $[X^*]$ .

The set of stable assignments describes the pattern of micro-matching between workers and machines with *manifest* technologies. Other assignments represent *latent* technologies, which are associated with outside alternatives to currently stable assignments.

### 3 Production Core, Technical Progress and Distribution

We define the concept of equilibrium using *production core*, represented by the set of stable factor-return distributions  $[X^*]$  that correspond to the set of stable assignments  $[\mu^*]$ . Since  $[\mu^*]$  summarizes all manifest production activities,  $V(\mu^*)$  measures the *GNP* of the microeconomy  $\mathcal{V}$  from the production side:

$$V(\mu^*) = \sum_{(i,j) \in [\mu^*]} v_{ij} \quad (5)$$

By measuring GNP from the income side based on stable factor distributions, we have:

$$V(\mu^*) = \sum_{i \in L} w_i + \sum_{j \in K} z_j \quad (6)$$

The considerations of technical progress do not change the fact that production core in our microeconomy contains the solution of the underlying linear assignment problem. Applying the von Neumann-Birkhoff duality theorem, one can focus on the dual concerning the distribution of factor returns and then prove the non-emptiness of production core, in terms of both equilibrium factor-return distributions in the dual problem and stable task assignments in the primal problem.

**Lemma 1:**  $[\mu^*] \neq \emptyset$  and  $[X^*] \neq \emptyset$ .

**Proof:** See Dantzig (1963), Shapley and Shubik (1972), and a sharper proof provided by Roth and Sotomayor (1990, Sections 8.1 and 8.2).

We next define the concept of technical progress. Under our general micro-matching framework, it is crucial to differentiate “Harrod/Hicks neutral type technical progress” from the “Harrod/Hicks neutral factor-return redistributions” wherein a particular type of neutral technical progress *need not* induce a neutral factor-return redistribution as established in neoclassical production theory. As a result of this inconsistency, we must now carefully differentiate the various types of neutral

technical progress based on not only the outcome of factor-return redistribution (see Definition 5) but the origin of innovation (see Definition 4).

Throughout this paper, we use the notation “tilde” to denote the post-technical progress entity.

**Definition 4:** Consider technical progress of size  $\lambda > 1$ . It is called

(i) **overall uniform** if  $\tilde{v}_{ij} = \lambda v_{ij}$  for all  $i$  and  $j$ ;

(ii) **labor- $i$  uniform** if  $\tilde{v}_{ij} = \lambda v_{ij}$  for all  $j$  and  $\tilde{v}_{i'j} = v_{i'j}$  for all  $i' \neq i$  and for all  $j$ .

Overall uniform technical progress can be viewed as generalization of neutral technical progress discussed by Hicks (1932) that features “shifts in production function over time by a uniform upward displacement of the entire function.” Labor- $i$  uniform technical progress can be regarded as a generalization of neutral technical progress considered by Harrod (1939) that features an “all-round increase in labor productivity” (*i.e.*, labor-augmenting). The origin of technical progress for the former case results from improvements in all workers and machines, whereas that for the latter is due exclusively to worker  $i$ .

**Remark 1:** It is clear that one may also define **machine- $j$  uniform** technical progress in the sense that  $\tilde{v}_{ij} = \lambda v_{ij}$  for all  $i$  and  $\tilde{v}_{ij'} = v_{ij'}$  for all  $j' \neq j$  and for all  $i$ . This type of technical progress resembles the capital-augmenting type proposed by Solow (1957). Because machine- $j$  uniform technical progress is mathematically isomorphic to labor- $i$  uniform technical progress if one interchanges indexes  $i$  and  $j$ , we will not discuss the machine- $j$  uniform case further but simply note that our results concerning labor- $i$  uniform technical progress will immediately apply to this case.

Following any type of technical progress, it is said that a **turnover** occurs if the post-technical progress stable assignment differs from the pre-technical progress stable assignment. Because we are interested in the timing of turnovers, it is necessary to specify the entire dynamic process of technical progress. Following R&D and innovation literature (e.g., see Aghion and Howitt 1992), consider that technical progress arrives at a Poisson rate  $\eta > 0$  with a scaling factor  $Z > 1$ . That is, technology improves by a factor  $Z$  over an average length of period  $1/\eta$ . Then, letting  $g = \frac{\dot{\lambda}(t)}{\lambda(t)}$  denote the rate of technical progress, we have:

$$\lambda(t) = \lambda(0)e^{gt}, \text{ with } g = \eta \ln(Z) > 0 \tag{7}$$



In the remainder of the paper, we shall assume, without loss of generality, that  $n = m$ . This is because, if, say,  $n > m$ , we can always add  $n - m$  dummy machines that yield zero payoffs into the microeconomy. It is evident that the state of the equilibrium is not changed by this act.

We now define the Hicks-neutral set and the Harrod-neutral set parallel to the concept used in neoclassical production theory.

**Definition 5:**

(i) Let  $\tilde{\mathcal{V}}$  be a microeconomy displaying overall uniform technical progress from an original microeconomy  $\mathcal{V}$ . The **Hicks-neutral set**  $\mathcal{K}[X]$  of  $[X]$  in  $\tilde{\mathcal{V}}$  is a set of technology-induced factor-return distributions given by,

$$\mathcal{K}[X] = \{(\tilde{w}_1, \dots, \tilde{w}_n, \tilde{z}_1, \dots, \tilde{z}_n) \in [\tilde{X}] \mid \text{there exists a factor-return distribution } (w_1, \dots, w_n, z_1, \dots, z_n) \in [X] \text{ such that } \frac{\tilde{w}_i}{w_i} = \frac{\tilde{z}_j}{z_j} \text{ for all } i, j = 1, \dots, n\}.$$

The **equilibrium Hicks-neutral set** is:  $\mathcal{K}^*(\mathcal{V}, \lambda) = \mathcal{K}([X^*])$ .

(ii) Let  $\tilde{\mathcal{V}}$  be a microeconomy displaying labor- $i$  uniform technical progress from an original microeconomy  $\mathcal{V}$ . The **Harrod-neutral set**  $\mathcal{H}_i[X]$  of  $[X]$  with respect to worker  $i$  in  $\tilde{\mathcal{V}}$  is a set of technology-induced factor-return distributions given by,

$$\mathcal{H}_i[X] = \{(w_1, \dots, w_{i-1}, \tilde{w}_i, w_{i+1}, \dots, w_n, z_1, \dots, z_n) \in [\tilde{X}] \mid \text{there exists a factor-return distribution } (w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n, z_1, \dots, z_n) \in [X] \text{ such that } \tilde{w}_i > w_i\}.$$

The **equilibrium Harrod-neutral set** is  $\mathcal{H}_i^*(\mathcal{V}, \lambda) = \mathcal{H}_i([X^*])$ .

By definition, Hicks neutrality must satisfy  $\tilde{w}_i/w_i = \tilde{z}_j/z_j = \lambda > 1$  for all  $i$  and  $j$ .

**Remark 2:** The concept of neutral technical progress therefore generalizes the neoclassical case illustrated by Allen (1938). These technological changes may even be of the learning-by-doing type with specific innovators as elaborated by Clemhout and Wan (1970).

Before turning to the subsequent section where we will establish useful properties associated with overall uniform technical progress and labor- $i$  uniform technical progress, we would like to provide a 2-by-2 example to illustrate the working of our two-sided matching framework.

**Example:** Consider a 2-by-2 microeconomy with  $v_{11} = 5$ ,  $v_{12} = 3$ ,  $v_{21} = 7$ , and  $v_{22} = 6$ . Under labor-2 uniform technical progress with  $\lambda > 1$ , the stable assignment becomes  $\mu^* = \{(1, 1), (2, 2)\}$ . The vertices of the set of stable distributions of factor returns can be derived in Table 1 below.<sup>7</sup>

Table 1: Stable Factor-Return Distributions

$w_1$	$w_2$	$z_1$	$z_2$	vertex
3	6	2	0	A
4	6	1	0	B
0	2	5	4	C
0	3	5	3	D

The two-dimensional projection of the set of stable factor-return distributions in  $(w_1, w_2)$  space is plotted in Figure 1. When there is weak labor-2 uniform technical progress with  $1 < \lambda < 8/7$ , the incremental returns to the innovating worker  $l_2$  are low compared to his/her created value of production. When labor-2 uniform technical progress is moderately strong with  $8/7 < \lambda < 2$ , there are no turnovers but the incremental returns to the innovating worker  $l_2$  now exceeds his/her created value of production as a result of his/her increasing threat of breaking the current match. When labor-2 uniform technical progress is sufficiently large with  $\lambda > 2$ , the previously stable assignment  $\mu^* = \{(1, 1), (2, 2)\}$  is destroyed and a new stable assignment  $\tilde{\mu}^* = \{(1, 2), (2, 1)\}$  is created.

## 4 Equilibrium Analysis

In this section, we will characterize how various types of technological advancements may influence micro-matching and distribution of factor returns and how technical progress may destroy an existing stable assignment and create a new stable assignment.

### 4.1 Overall Uniform Technical Progress

Consider a microeconomy  $\tilde{\mathcal{V}}$  that displays overall uniform technical progress from an original microeconomy  $\mathcal{V}$ . As there will be no turnover after overall uniform technical progress, we have:

**Theorem 1:** (Stable Assignments and Equilibrium Hicks-Neutral Set) *Under overall uniform technical progress,  $[\tilde{\mu}^*] = [\mu^*]$  and  $\mathcal{K}^*(\mathcal{V}, \lambda) = [\tilde{X}^*]$ .*

<sup>7</sup>The solution technique used herein is the Hungarian algorithm, as in Dantzig (1963) and Sharpley and Shubik (1972). An alternative is to adopt the deferred acceptance algorithm developed by Crawford and Knoer (1981).

*Proof:* After overall uniform technical progress, each side of (3) and (4) will be multiplied by  $\lambda$ , which can be all cancelled out and reduced to the original forms as given by (3) and (4). Thus, the set of stable assignments remain unchanged. By the definition of overall uniform technical progress,  $\frac{\tilde{w}_i}{w_i} = \frac{\tilde{z}_j}{z_j} = \lambda$ . From the definition of  $\mathcal{K}^*(\mathcal{V}, \lambda)$  and expressions (3) and (4) augmented by  $\lambda$ , the second result follows immediately. ■

## 4.2 Labor- $i$ Uniform Technical Progress

In contrast with the case of overall uniform technical progress, this analysis of the consequences of labor- $i$  uniform technical progress is much more complex as a result of the possibility of turnover when the size technical progress  $\lambda$  is large.

Denote  $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{2n})$  and the distribution before and after technical progress to all factors but the innovator (labor- $i$ ) as  $[X_{-i}^*]$  and  $[\tilde{X}_{-i}^*]$ . Changing matches as a result of turnover form a **chain** (or, a “circle” of modulus  $n$  without considering the ordering), denoted by  $C$ . Call those players involved in these changing matches as **chain players**, with the set of chain players denoted by  $A_c$ ; thus, the set  $A \setminus A_c = L \cup K \setminus A_c$  contains all non-chain players (obviously, all dummy agents added to the microeconomy with zero payoffs must be non-chain players). Thus, for non-chain players, there is no turnover of their task assignments. This decomposition enables us to focus on establishing properties concerning mainly chain players.

We begin by proving that the equilibrium Harrod-neutral set is non-empty. Its non-emptiness follows from the fact that a stable assignment exists both before and after the turnover (Lemma 1) and that payoffs are higher after the turnover and hence more returns can be distributed to production factor inputs.

**Theorem 2:** (Equilibrium Harrod-neutral Set)  $\mathcal{H}_i^*(\mathcal{V}, \lambda) \neq \phi$ .

*Proof:* We consider two subcases.

- (i) No turnover  $[\tilde{\mu}^*] = [\mu^*]$ : The nonemptiness of the equilibrium Harrod-neutral set can be proved by construction by assigning  $\tilde{x}_i = x_i + \Delta_i(x)$  with  $\Delta_i(x) = V(\mu^*) - \sum_{a \neq i} x_a - x_i$ .
- (ii) Turnover  $[\tilde{\mu}^*] \neq [\mu^*]$ : Let  $x = (x_1, \dots, x_j, \dots, x_{\tilde{j}}, \dots, x_{2n})$ . For any  $x \in [X^*]$ , construct  $\bar{x}$  in such a way that  $\bar{x}_{-i} = x_{-i}$  and  $\bar{x}_i = x_i + \Delta_i(x)$ , where  $\bar{x}_{-i} \in Q_{-i} = \{x_{-i} | x_{-i} \in [x_{-i}^*] \cap [\tilde{x}_{-i}^*]\}$ . Let

$$Q = \{\bar{x} | \bar{x}_{-i} \in Q_{-i}, \bar{x}_i = v(\tilde{\mu}^*) - \bar{x}_{-i} \cdot 1_{2n-1}^T\}, \quad (8)$$

where  $1_{2n-1}^T$  is the transpose of a vector of  $2n-1$  ones. Then, it is clear that by construction  $Q \neq \phi$ . But by definition,  $Q = \mathcal{H}_i^*(\mathcal{V}, \lambda)$ , implying  $\mathcal{H}_i^*(\mathcal{V}, \lambda) \neq \phi$ . ■

Turnover can never occur under overall uniform technical progress. Labor- $i$  uniform technical progress may result in turnover, destroying existing stable assignments and creating new stable assignments. The set of equilibrium Hicks-neutral factor-returns distributions of under overall uniform technical progress is always identical to the new core. The set of equilibrium Harrod-neutral factor-returns distributions under labor- $i$  uniform technical progress is non-empty.

We can further characterize turnovers and equilibrium distributions in the following theorems.

To simplify the illustration, we reorder all agents in such a way that  $\ell_i$  and  $k_{j(i)}$  are matched prior to labor- $i$  overall uniform technical progress of size  $\lambda > 1$  and  $\ell_i$  and  $k_{j(i)+1}$  are matched after turnover (a circle of modulus  $n$ ). Without loss of generality, we relabel  $j(i) = i$  and delineate the micro-matching for chain and non-chain players in Table 2.

Table 2: Chain and Non-Chain Players After a Turnover

	$\ell_1$	$\ell_2$	$\cdots$	$\ell_{i-1}$	$\ell_i$	$\ell_{i+1}$	$\cdots$	$\ell_s$	$\ell_{s+1}$	$\cdots$	$\ell_n$
$k_1$	$(1,1)$ $\in[\mu^*]$							$(s,1)$ $\in[\tilde{\mu}^*]$			
$k_2$	$(1,2)$ $\in[\tilde{\mu}^*]$	$\cdots$									
$\vdots$		$\cdots$	$\cdots$								
$k_{i-1}$			$\cdots$	$(i-1,i-1)$ $\in[\mu^*]$							
$k_i$				$(i-1,i)$ $\in[\tilde{\mu}^*]$	$(i,i)$ $\in[\mu^*]$						
$k_{i+1}$					$(i,i+1)$ $\in[\tilde{\mu}^*]$	$(i+1,i+1)$ $\in[\mu^*]$					
$\vdots$						$\cdots$	$\cdots$				
$k_s$							$\cdots$	$(s,s)$ $\in[\mu^*]$			
$k_{s+1}$									$(s+1,s+1)$ $\in[\mu^*] \cap [\tilde{\mu}^*]$		
$\vdots$										$\cdots$	
$k_n$											$(n,n)$ $\in[\mu^*] \cap [\tilde{\mu}^*]$

Here, each agent is labelled by an element in the index set  $I = \{1, 2, \dots, n\}$ . The entry of the diagonal,  $(1, 1), \dots, (i, i), \dots, (s, s)$ , are elements of  $[\mu^*]$ , whereas the entry,  $(1, 2), \dots, (i, i+1), \dots, (s, 1)$ , are elements of  $[\tilde{\mu}^*]$ . Thus, each chain player can now be indexed by an element in  $I_c = \{1, 2, \dots, s\} \subset I$ . For the non-chain players,  $\ell_{s+1}, \dots, \ell_n, k_{s+1}, \dots, k_n$ , their equilibrium matches remain unchanged.

We then determine explicitly when a turnover would occur:

**Theorem 3:** (Critical Point and Critical Time of Turnover) *When a turnover occurs under labor- $i$  uniform technical progress of size  $\lambda > 1$ , it must satisfy:*

$$\lambda > \Lambda_i(C) = \frac{\sum_{a \neq i, a \in I_c} (v_{a,a} - v_{a,a+1})}{v_{i,i+1} - v_{i,i}} \quad (9)$$

*Under the dynamic process specified in (7) with  $\lambda(0) = 1$ , the critical calendar time for turnover to occur is given by,*

$$T_i(C) = \frac{\ln \left[ \sum_{a \neq i, a \in I_c} (v_{a,a} - v_{a,a+1}) / (v_{i,i+1} - v_{i,i}) \right]}{\eta \ln(Z)} \quad (10)$$

*Proof:* Applying the GNP equation (5) both before and after the turnover, we have:

$$V(\mu^*) = \sum_{a \in I} v_{aa} = \sum_{a \in I_c} v_{aa} + \sum_{a \in I \setminus I_c} v_{aa} \quad (11)$$

$$V(\tilde{\mu}^*) = \sum_{a \in I} \tilde{v}_{a,a+1} = \sum_{a \neq i, a \in I_c} v_{a,a+1} + \sum_{a \in I \setminus I_c} v_{a,a+1} + \lambda v_{i,i+1} \quad (12)$$

Equating  $V(\tilde{\mu}^*)$  with  $V(\mu^*)$  yields the critical value  $\Lambda_i(C)$  as specified in (9). Substituting the equality in (9) into (7) yields (10). ■

One may regard  $1/T_i(C)$  as a measure of the “speed of turnover” – a larger value means a turnover can occur even with small sized labor- $i$  uniform technical progress. Theorem 3 indicates that the speed of turnover depends crucially on the arrival rate and scaling factor of technical progress, as well as the relative productivity of the matches.

**Remark 3:** While it is straightforward that both the arrival rate and scaling factor of technical progress ( $\eta$  and  $Z$ ) raise the speed of turnover, the effect of the relative productivity of the matches on the speed of turnover deserves further illustration.

- (i) The term  $v_{i,i+1} - v_{i,i}$  is the incremental value accrued for the innovating worker  $\ell_i$  to give up the existing match and to subsequently create a chain of new matches; it can thus be viewed as a measure of the temptation for the innovating worker to alter the match.
- (ii) The term  $\sum_{a \neq i, a \in I_c} (v_{a,a} - v_{a,a+1})$  summarizes the aggregate opportunity costs of separating the existing matches of other chain players, which measures the level of resistance to new matches.

When the temptation for the innovating worker to alter the match is relatively high compared to the level of resistance to rematches, turnover can occur with a relatively small critical size  $\Lambda_i(C)$  and relatively shorter critical calendar time.

Upon examining the critical point of turnover, we now turn to studying the properties of the equilibrium Harrod-neutral set under our general micro-matching setup (Theorems 4 and 5) and the neoclassical counterpart of Harrod-neutral distributions (Theorem 6).

**Theorem 4:** (Equilibrium Distribution After labor- $i$  uniform Technical Progress)  $\forall a \in I_c$ , index  $(a, j(a)) \in [\mu^*]$ ,  $(a, j(a)+1) \in [\tilde{\mu}^*]$ . Relabel  $j(a) = a$  and define  $\Delta w_i = \tilde{w}_i - w_i$  and  $\Delta v_{i,j} = \tilde{v}_{i,j} - v_{i,j}$ . Let  $(w, z) \in [X^*]$ ,  $(\tilde{w}, \tilde{z}) \in [\tilde{X}^*]$  and  $|A_c| = s$ . Then,

(i) the equilibrium distribution to workers satisfies:

$$\Delta w_{i-1} + \Delta v_{ii} \leq \Delta w_i \leq \Delta w_{i+1} + \Delta v_{i,i+1} \quad (13)$$

$$\Delta w_s \leq \Delta w_1 \quad \text{and} \quad \Delta w_{a-1} \leq \Delta w_a \quad \text{for all } a \neq i \text{ or } i+1 \quad (14)$$

(ii) the equilibrium distribution to machines satisfies:

$$\Delta z_{i+1} - \Delta z_i \leq \Delta v_{i,i+1} - \Delta v_{ii} \quad (15)$$

$$\Delta z_1 \leq \Delta z_s \quad \text{and} \quad \Delta z_{a+1} \leq \Delta z_a \quad \text{for all } a \neq i \quad (16)$$

*Proof:* To prove part (i), we use (3) and (4) to write,

$$w_i + z_{j(i)} = v_{i,j(i)}, \quad w_i + z_{j(i)+1} \geq v_{i,j(i)} \quad (17)$$

$$\tilde{w}_i + \tilde{z}_{j(i)+1} = \tilde{v}_{i,j(i)+1}, \quad \tilde{w}_i + \tilde{z}_{j(i)} \geq \tilde{v}_{ij} \quad (18)$$

By eliminating  $z_{j(i)}$  and  $\tilde{z}_{j(i)+1}$ , using  $z_j = v_{ij} - w_i$  and  $\tilde{z}_j = \tilde{v}_{i-1,j} - \tilde{w}_{i-1}$  and relabelling  $j(i) = i$ , (17) and (18) imply:

$$w_i - w_{i-1} \leq v_{ii} - v_{i-1,i} \quad (19)$$

$$w_{i+1} - w_i \leq v_{i+1,i+1} - v_{i,i+1} \quad (20)$$

$$\tilde{w}_i - \tilde{w}_{i-1} \geq \tilde{v}_{ii} - v_{i-1,i} = \lambda v_{ii} - v_{i-1,i} = v_{ii} - v_{i-1,i} + \Delta v_{ii} \quad (21)$$

$$\tilde{w}_{i+1} - \tilde{w}_i \geq \tilde{v}_{i+1,i+1} - \tilde{v}_{i,i+1} = v_{i+1,i+1} - \lambda v_{i,i+1} = v_{i+1,i+1} - v_{i,i+1} - \Delta v_{i,i+1} \quad (22)$$

Utilizing (19) and (21), one obtains the first inequality in (13); combining (20) and (22) further yields the second inequality in (13). We can then obtain (14) by applying (13) to any worker  $\ell_a \in A_c$  and by recognizing that  $\Delta v_{aa} = \Delta v_{a,a+1} = 0$  for  $a \neq i$ ,  $a \in I_c$  and that chain players is a circle of modulus  $s$ .

Similarly, we can prove part (ii) by using (3) and (4) to obtain:

$$w_i + z_{j(i)} = v_{i,j(i)}, \quad w_{i-1} + z_{j(i)} \geq v_{i-1,j(i)} \quad (23)$$

$$\tilde{w}_{i-1} + \tilde{z}_{j(i)} = \tilde{v}_{i-1,j(i)}, \quad \tilde{w}_i + \tilde{z}_{j(i)} \geq \tilde{v}_{ij(i)} \quad (24)$$

By eliminating  $w_i$  and  $\tilde{w}_i$  and relabelling  $j(i) = i$ , (23) and (24) imply, respectively,

$$z_i - z_{i-1} \geq v_{i-1,i} - v_{i-1,i-1} \quad (25)$$

$$z_{i+1} - z_i \geq v_{i,i+1} - v_{ii} \quad (26)$$

$$\tilde{z}_{i-1} - \tilde{z}_i \geq \lambda(v_{i-1,i-1} - v_{i-1,i}) \quad (27)$$

$$\tilde{z}_i - \tilde{z}_{i+1} \geq \lambda(v_{ii} - v_{i,i+1}) \quad (28)$$

We can now combine (25) and (27) to get the inequality in (15) and combine (26) and (28) to yield an inequality in (16):  $\Delta z_i \leq \Delta z_{i-1}$ . We finally prove the remaining inequalities in (16) by applying (15) to  $a \in I_c$  and by recognizing that  $\Delta v_{aa} = \Delta v_{a,a+1} = 0$  for  $a \neq i$ . ■

The properties derived in Theorem 4 go far beyond the classic neoclassical production theory. Here, we establish not only how labor-augmenting technical progress enhances the innovating worker's return, but also how it influences (i) the innovating worker's old and new mates (machines  $k_i$  and  $k_{i+1}$ , respectively), (ii) the innovating worker's direct competitors (worker  $\ell_{i-1}$ , who takes over  $\ell_i$ 's pre-turnover matching machine  $k_i$ , and worker  $\ell_{i+1}$ , who yields his/her pre-turnover matching machine  $k_{i+1}$  to  $\ell_i$  after the turnover), and (iii) the innovating worker's indirect competitors (all other workers in the chain  $\ell_a$ ,  $a \in I_c \setminus \{i-1, i, i+1\}$ ).

**Remark 4:** Theorem 4 establishes the spillover effects of technical progress pertaining to worker  $\ell_i$ .

(i) (Changes in the Returns to Non-innovating Workers) The first inequality of (13) implies  $\Delta w_i - \Delta w_{i-1} \geq \Delta v_{ii}$ . That is, after a turnover, the incremental return to the innovating worker  $\ell_i$  exceeds that to worker  $\ell_{i-1}$  (who takes over  $\ell_i$ 's pre-turnover matching machine) by at least

the incremental value of production accrued to  $i$ 's *pre-turnover* stable assignment  $(i, i)$ . The second inequality of (13) says  $\Delta w_i - \Delta w_{i+1} \leq \Delta v_{i,i+1}$ . Thus, after a turnover, the incremental return to the innovating worker  $\ell_i$  exceeds that to worker  $\ell_{i+1}$  (who yields his/her matching machine to  $\ell_i$  after the turnover) by no more than the incremental value of production accrued to  $\ell_i$ 's *post-turnover* stable assignment  $(i, i+1)$ . The worker yielding his/her pre-turnover mate to  $\ell_i$  after the turnover (worker  $\ell_{i+1}$ ) is the “head” of the chain whereas the worker taking over  $\ell_i$ 's pre-turnover mate (worker  $\ell_i$ ) is the “tail” of the chain. Because the former suffers the most direct loss from turnover (directly crowded out by the innovating worker), his/her incremental return must be less than those less directly influenced at a later position of the chain  $C$ . This gives the entire ordering of incremental returns specified in (14): the incremental returns to workers as a result of turnover increases along the chain.

- (ii) (Changes in the Returns to Machines) The value of production differential in (15),  $\Delta v_{i,i+1} - \Delta v_{ii}$ , is always positive (otherwise, turnover would have not occurred). The innovating worker  $\ell_i$ 's new mate (machine  $k_{i+1}$ ) is the head of the chain whereas  $\ell_i$ 's old mate (machine  $k_i$ ) is the tail of the chain. Being the innovating worker's new mate would receive the greatest benefits. Thus, the incremental returns to machines as a result of turnover decreases along the chain, as given by (16). Such a redistribution must, however, be limited by the extent of technical progress. As a consequence, the differential between the incremental returns to the machine in the head position (machine  $k_{i+1}$ ) and that to the one in the tail position (machine  $k_i$ ) is bounded by the differential in the value of production between the new stable assignment  $(i, i+1)$  and the old one  $(i, i)$  (*i.e.*,  $\Delta v_{i,i+1} - \Delta v_{ii}$ ).

The proof of Theorem 4 also generates a useful implication concerning how turnovers may change the dimensionality of the equilibrium Harrod-neutral set. In particular, it shows that even when the production core is of full dimension before and after labor- $i$  uniform technical progress with  $\dim[X^*] = \dim[\tilde{X}^*] = n + m = 2n$ , turnovers can cause a reduction in  $\dim[\mathcal{H}_i^*(\mathcal{V}, \lambda)]$ .

**Theorem 5:** (Reduction in Dimensionality of Equilibrium Harrod-Neutral Set) *Consider labor- $i$  uniform technical progress of size  $\Lambda_i(C)$ . Let  $[X^*]$  and  $[\tilde{X}^*]$  both be of full dimension. Then,*

$$\dim[\mathcal{H}_i^*(\mathcal{V}, \lambda)] = \dim[X^*] - \left( \frac{|A_c|}{2} - 1 \right) \quad (29)$$

*Proof:* From (17) and (18) in the proof of Theorem 4, the dimensionality reduces by one from one



pair to another pair of chain players. Thus, the dimensionality of the equilibrium Harrod-neutral set shrinks by  $\frac{|A_c|}{2} - 1$ . ■

Finally, we would like to contrast our results with findings in neoclassical production theory. To facilitate such comparison, we select redistribution of factor returns after labor- $i$  uniform technical progress in such a way to satisfy  $\tilde{w}_a = w_a$  for each  $a \neq i$ ,  $a \in I_c$  and  $\tilde{z}_a = z_a$  for each  $a \in I_c$  (consistent with the conventional neoclassical Harrod-neutral distribution). By the proof of Theorem 4, it is not difficult to find a stable distribution of factor returns such that  $\tilde{w}_i > w_i$ . Thus, our results *contain* those in neoclassical production theory. In this special case, we can further pin down explicitly the incremental return to the innovating worker.

**Theorem 6:** (Equilibrium Neoclassical Harrod-neutral Distribution) *For  $a \in I_c$ , index  $(a, j(a)) \in [\mu^*]$ ,  $(a, j(a) + 1) \in [\tilde{\mu}^*]$ . Relabel  $j(a) = a$  and define  $\Delta w_i = \tilde{w}_i - w_i$ ,  $\Delta v_{i,j} = \tilde{v}_{i,j} - v_{i,j}$ , and  $\Delta V = V(\tilde{\mu}^*) - V(\mu^*)$ . Let  $(w, z) \in [X^*]$  and  $(\tilde{w}, \tilde{z}) \in \mathcal{H}_i^*(V, \lambda)$ . A neoclassical Harrod-neutral distribution with  $\tilde{w}_a = w_a$  for each  $a \neq i$  and with  $\tilde{z}_a = z_a$  for each  $a$  satisfies:*

(i) (marginal productivity)

$$\Delta w_i = \Delta V \quad (30)$$

(ii) (incremental return to the innovating worker)

$$\Delta w_i = (\lambda - 1)v_{i,i+1} + [(v_{i-1,i} + v_{i,i+1}) - (v_{i,i} + v_{i+1,i+1})] - \sum_{a \in A_c, a \neq i, i-1} (v_{a+1,a+1} - v_{a,a+1}) \quad (31)$$

*Proof:* Set  $\tilde{w}_a = w_a$  ( $a \neq i$ ) and  $\tilde{z}_a = z_a$  and let  $j = i$  (by relabeling). Applying (6) both before and after the turnover, we have:

$$V(\mu^*) = \sum_{a \in L} w_a + \sum_{a \in K} z_a \quad \text{and} \quad V(\tilde{\mu}^*) = \tilde{w}_i + \sum_{a \in L, a \neq i} w_a + \sum_{a \in K} z_a \quad (32)$$

which can be combined to yield (30) in part (i).

To prove part (ii), we utilize (17), (21) and (22) to derive

$$w_{a+1} - w_a = v_{a+1,a+1} - v_{a,a+1} \quad \forall a \neq i \text{ or } i - 1, a \in I_c \quad (33)$$

Next, (17), (21) and (22) together yield:

$$(\Delta v_{ii} - \Delta w_i) + v_{ii} - v_{i-1,i} \leq w_i - w_{i-1} \leq v_{ii} - v_{i-1,i} \quad (34)$$

$$(\Delta v_{i,i+1} - \Delta w_i) + v_{i+1,i+1} - v_{i,i+1} \leq w_{i+1} - w_i \leq v_{i+1,i+1} - v_{i,i+1} \quad (35)$$

Applying (5) and (6) both before and after the turnover, we get:

$$V(\mu^*) = \sum_{a \in A_c} v_{aa} + \sum_{a \notin A_c} v_{aa} = \sum_{a \in L} w_a + \sum_{a \in K} z_a \quad (36)$$

$$V(\tilde{\mu}^*) = \sum_{a \in A_c, a \neq i} v_{a,a+1} + \sum_{a \notin A_c} v_{a,a+1} + \lambda v_{i,i+1} = \tilde{w}_i + \sum_{a \in L, a \neq i} w_a + \sum_{a \in K} z_a \quad (37)$$

These can then be combined with (33)-(35) to obtain (31). ■

Theorem 6 delivers two sharp results. The first is on marginal productivity, illustrating the absorption of the microeconomy-wide productivity gain by the innovating worker. The second further solves explicitly incremental returns to the innovating worker, which need not be equal to the incremental value of production from the innovating worker's post-turnover new match.

**Remark 5:** The results established in Theorem 6 deserve further comments.

(i) (Marginal Productivity) One may regard part (i) of Theorem 6 as a special form of marginal productivity theory in the context of the neoclassical Harrod-neutrality distribution that resembles the “no-surplus” condition in general equilibrium theory (cf. Ostroy 1980). Its meaning is straightforward. As a result of worker  $\ell_i$ 's innovation, the aggregate surplus accrued in the microeconomy is  $\Delta V$ . When this microeconomy-wide gain is completely internalized by the innovating worker ( $\Delta w_i = \Delta V$ ), there must be no surplus accrued to the remaining agents (i.e.,  $\tilde{w}_a = w_a$  for each  $a \neq i$ ,  $a \in I_c$  and  $\tilde{z}_a = z_a$  for each  $a \in I_c$ ). Conversely, when the no-surplus condition holds, the microeconomy-wide gain must be fully absorbed by the innovating worker.

(ii) (Incremental Returns to the Innovating Worker) Part (ii) of Theorem 6 suggests that labor- $i$  uniform technical progress benefits the innovating worker  $\ell_i$  only when

- a. such labor-augmenting technical progress is sizable (i.e.,  $\lambda$  is large),
- b. the value of production associated with the new match is sufficiently higher than that with the old match (i.e.,  $(v_{i-1,i} + v_{i,i+1}) - (v_{i,i} - v_{i+1,i+1})$  is large),
- c. the change in the value of production from rematches for other chain players is sufficiently low (i.e.,  $\sum_{a \in A_c, a \neq i, i-1} (v_{a+1,a+1} - v_{a,a+1})$  is small).

Purely from the viewpoint of the post-turnover manifest technology associated with the innovating worker, the incremental return to the innovating worker may be greater than or less than

the direct incremental value of production created by the manifest technology associated with the post-turnover stable task assignment  $(i, i+1)$ . That is,  $\Delta w_i - \Delta v_{i,i+1} = \Delta w_i - (\lambda - 1)v_{i,i+1}$  may be positive or negative, in contrast with its counterpart in neoclassical production theory where incremental return to the innovating worker must be equal to the direct incremental value of production created by the manifest technology associated with the innovating worker. This different finding results from two special features of our two-sided micro-matching framework. One is the explicit account for the role of latent technologies as outside alternatives. Another is the explicit account for the spillovers from the innovating worker to his/her potential mates and his/her directly and indirectly competing workers. Thus, even under the neoclassical Harrod-neutral distribution scheme, our results are much richer than those obtained in neoclassical production theory.

### 4.3 The Special Case of Two-by-Two

To gain further insights, let us consider a microeconomy with two workers and two machines, *i.e.*,  $n = m = 2$ . In this case, should a turnover occur as a result of labor- $i$  uniform technical progress, all agents must be chain players. Without loss of generality, let us set:

$$v_{21} \geq v_{22} \geq v_{11} \geq v_{11} + v_{22} - v_{21} \geq v_{12}$$

In this case, the efficient assignment is  $[\mu^*] = \{(1, 1), (2, 2)\}$ . Under a labor-2 uniform technical progress of size  $\lambda$ , one can compute the critical value as:

$$\Lambda_2(C) = (v_{11} - v_{12}) / (v_{21} - v_{22})$$

Under the dynamic process specified in (7) with  $\lambda(0) = 1$ , the critical calendar time for turnover to occur becomes:

$$T_2(C) = \ln \left( \frac{v_{11} - v_{12}}{v_{21} - v_{22}} \right) / [\eta \ln(Z)]$$

Consider two cases, one without turnover ( $\lambda = \lambda_N$ ) and one with turnover ( $\lambda = \lambda_T$ ), with the size of technical progress satisfying,

$$\lambda_N < \Lambda_2(C) < \lambda_T$$

>From (3) and (4), we obtain the equilibrium distributions of factor returns prior to technical progress and report the results in Table 3 below.

Table 3: Equilibrium Factor-Return Distributions

$w_1$	$w_2$	$z_1$	$z_2$	vertex
$v_{12}$	$v_{22}$	$v_{11} - v_{12}$	0	A
$v_{11} + v_{22} - v_{21}$	$v_{22}$	$v_{21} - v_{22}$	0	B
0	$v_{21} - v_{11}$	$v_{11}$	$v_{11} + v_{22} - v_{21}$	C
0	$v_{22} - v_{12}$	$v_{11}$	$v_{12}$	D

From (4) and the nonnegativity constraints of factor returns, it is trivial that both  $w_1$  and  $z_1$  are within the range of  $[0, v_{11}]$ , whereas  $w_2$  and  $z_2$  are in  $[0, v_{22}]$ . Moreover, (3) gives additional boundaries for the distributions of factor returns. We plot the two-dimensional projections of the (four-dimensional) set of stable factor-return distributions,  $[X^*]$ , onto  $(w_1, w_2)$  and  $(z_1, z_2)$  space, respectively; see sets  $ABCD$  in Figures 2 and 3.

With technical progress, the equilibrium Harrod-neutral set in this case can be rewritten as:

$$\mathcal{H}_2^*(\mathcal{V}, \lambda) = \{(w_1, \tilde{w}_2, z_1, z_2) \in [\tilde{X}^*] \mid \text{there exists } (w_1, w_2, z_1, z_2) \in [X^*] \text{ such that } \tilde{w}_2 > w_2\}$$

The projections of the  $\mathcal{H}_2^*(\mathcal{V}, \lambda)$  onto  $(w_1, w_2)$  and  $(z_1, z_2)$  space, respectively, are also plotted in Figures 2 and 3. Since all agents are chain players,  $|A_c| = 4$ . Projections of  $\mathcal{H}_2^*(\mathcal{V}, \lambda_N)$ , the case without turnover, onto  $(w_1, w_2)$  and  $(z_1, z_2)$  space, are sets  $A'B'C'D'$  in Figures 2 and 3, respectively, where  $\mathcal{H}_2^*(\mathcal{V}, \lambda_N)$  has a full dimension of 4. Projections of  $\mathcal{H}_2^*(\mathcal{V}, \lambda_T)$ , the case with turnover, onto  $(w_1, w_2)$  and  $(z_1, z_2)$  space, are set  $A''B''C''D''$  in Figure 2 and line segment  $A''D''$  in Figure 3, respectively. The dimensionality of  $\mathcal{H}_2^*(\mathcal{V}, \lambda_T)$  reduces from 4 to 2, thus verifying (29).

**Example:** Consider the 2-by-2 microeconomy given in Section 3 with  $v_{11} = 5$ ,  $v_{12} = 3$ ,  $v_{21} = 7$ , and  $v_{22} = 6$ . The critical value of turnover is  $\Lambda_2(C) = 2$ . If such labor-2 uniform technical progress arrives twice a year ( $\eta = 2$ ) with an expansion rate of 5% ( $Z = 1.05$ ), then worker  $\ell_2$  improves at an annual rate of  $2(\ln 1.05) \approx 9.76\%$  and turnover will occur after  $\frac{\ln 2}{2(\ln 1.05)} \approx 7.1$  years.

## 5 Concluding Remarks

We have constructed a two-sided micro-matching framework with heterogeneous workers and machines, allowing for a very general production technology without *a priori* restrictions on the functional form. We have established some punch-line properties.

We find that turnover can never occur under overall uniform technical progress and that labor- $i$  uniform technical progress may result in turnover, destroying existing stable assignments and

creating new ones. The equilibrium set of Hicks-neutral factor-return distributions under overall uniform technical progress is always identical to the new core, but the equilibrium set of Harrod-neutral factor-return distributions may not be. Labor- $i$  uniform technical progress may not create turnover if the size of technical progress is sufficiently small, or if the resistance from the existing matches is strong, relative to the innovator's productivity gain.

After a turnover as a result of labor- $i$  uniform technical progress, the incremental return to the innovating worker  $\ell_i$  (i) exceeds that to the worker who takes over  $\ell_i$ 's pre-turnover matching machine by at least the incremental value of production accrued to  $\ell_i$ 's pre-turnover match and (ii) exceeds that to the worker who yields his/her matching machine to  $\ell_i$  after the turnover by no more than the incremental value of production accrued to  $\ell_i$ 's post-turnover match. Under the neoclassical Harrod-neutral distribution scheme, the innovating worker acquires the entire microeconomy-wide gain, which may be greater or less than the direct incremental value of production created by the manifest technology associated with the innovating worker's new match. In general, labor- $i$  uniform technical progress creates spillovers in factor-return distributions to all other agents as a result of turnover. On the one hand, the incremental returns to other workers increase along the chain, ordered from the worker who yields his/her matching machine to  $\ell_i$  to the worker who takes over  $\ell_i$ 's pre-turnover matching machine. On the other hand, the incremental returns to machines decrease along the chain, ordered from  $\ell_i$ 's new mate to  $\ell_i$ 's old mate.

Along these lines, one may study the equilibrium consequences of a non-neutral technical progress, which may involve only a single task or be localized to a subset of agents. It is interesting to characterize return-spillovers to non-innovating players. A second avenue that may be of interest is to examine another main issue of dynamics, namely, factor accumulation. In the two-sided micro-matching microeconomy, accumulation of a particular factor can be viewed as entry of an identical twin of a particular agent. The basic methodology established in this paper is, with the assistance from the setup in the entry game by Mo (1988), readily applied to this extension. Versions of magnification properties such as Jones-Rybczynski theorem may then be established in response to the expansion of a particular factor. A third avenue is to generalize the one-to-one matching structure to many-to-one (cf. Kelso and Crawford 1982) or many-to-many (cf. Roth 1984). While the generalization to many-to-one matching may be useful for studying the behavior of firm with many workers, the generalization to many-to-many matching may be particularly relevant to understanding the interactions between producing firms and outsourcing subcontractors.

## References

- [1] Aghion, P. and P. Howitt (1992), "A Model of Growth Through Creative Destruction," *Econometrica*, 60, 323-351.
- [2] Allen, R. G. D. (1938), *Mathematical Analysis for Economists*, Macmillan, London, UK.
- [3] Arrow, K., H. B. Chenery, B. S. Minhas, and R. M. Solow (1961), "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, 43, 225-250.
- [4] Chen, B., J. Mo and P. Wang (2002), "Technology Adoption, Growth and Matching," *Journal of Economic Dynamics and Control*, 26, 1927-1954.
- [5] Clemhout, S. and H. Y. Wan, Jr. (1970), "Learning-by-Doing and Infant Industry Protection," *Review of Economic Studies*, 37, 33-56.
- [6] Crawford, V. P. (1988), "Comparative Statics in Matching Markets," *Journal of Economic Theory*, 54, 389-400.
- [7] Crawford, V. P. and E. M. Knoer (1981), "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49, 437-450.
- [8] Dantzig, G. (1963), *Linear Programming and Extensions*, Princeton University Press, Princeton.
- [9] Demange, G. and D. Gale (1985), "The Strategy Structure of Two-Sided Matching Markets," *Econometrica*, 53, 873-888.
- [10] Fallon, P. and R. Layard (1975), "Capital-Skill Complementarity, Income Distribution, and Output Accounting," *Journal of Political Economy*, 83, 279-302.
- [11] Gale, D. and L. Shapley (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9-15.
- [12] Hicks, J. R. (1932), *The Theory of Wages*, Macmillan, London, UK.
- [13] Harrod, R. F. (1939), "An Essay in Dynamic Theory," *Economic Journal*, 49, 14-33.
- [14] Kelso, A. S., J. and V. P. Crawford (1982), Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483-1504.
- [15] Krusell, P., L. Ohanian, J. Ríos-Rull, and G. L. Violante (2000), "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis," *Econometrica*, 68, 1029-1053.
- [16] Laing, D. , T. Palivos, and P. Wang (1995), "Learning, matching, and growth," *Review of Economic Studies*, 62, 115-129.
- [17] Marshall, A. (1890), *Principles of Economics: An Introductory Volume*, (1990 reprint of 1920 edition), Porcupine, Philadelphia, PA.

- [18] Mo, J., “Entry and Structures of Interest Groups in Assignment Games,” *Journal of Economic Theory*, 46, 66-96.
- [19] Ostroy, J. M. (1980), “The No-Surplus Condition as a Characterization of Perfectly Competitive Equilibrium,” *Journal of Economic Theory*, 22, 183-207.
- [20] Robinson, J. V. (1937), “Some Reflections on Marxist Economics,” *Economic Journal*, 328-329.
- [21] Roth, A. E. (1984), “Stability and Polarization of Interests in Job Matching,” *Econometrica*, 52, 47-57.
- [22] Roth, A. E. and M. A. Sotomayor (1990), *Two sided Matching: A study in Game-Theoretic Modelling and Analysis*, Cambridge University Press, Cambridge, MA.
- [23] Samuelson, P. A. (1962), “Parable and Realism in Capital Theory: The Surrogate Production Function,” *Review of Economic Studies*, 29, 193-206.
- [24] Sato, R. (1975), “The Most General Class of CES Function,” *Econometrica*, 43, 999-1003.
- [25] Shapley, L. and M. Shubik (1972), “The Assignment game I: The core,” *International Journal of Game Theory*, 1, 111-130.
- [26] Shimomura, K. (1999), “A Simple Proof of the Sato Proposition on Non-Homothetic CES Functions,” *Economic Theory*, 14, 501-503.
- [27] Solow, R. M. (1957), “Technical Change and the Aggregate Production Function,” *Review of Economics and Statistics*, 39, 312-320.
- [28] Uzawa, H. (1961), “Neutral Inventions and the Stability of Growth Equilibrium,” *Review of Economic Studies*, 28, 117-124.
- [29] Von Neumann, J. (1953), “A Certain Zero-Sum Two-Person Game Equivalent to the Optimal Assignment Problems,” in Harold W. Kuhn and Albert W. Tucker (eds), *Contributions to the Theory of Games* (Vol. 2), Princeton University Press, Princeton, NJ.
- [30] Wan, H. Y., Jr. (1971), *Economic Growth*, Harcourt, Brace & Jovanovich, New York, NY.
- [31] Walras, L. (1874), *Elements of Pure Economics: Or the Theory of Social Wealth*, (1954 translation of 1926 edition), Richard Irwin, Homewood, IL.
- [32] Wicksell, K. (1900), “Marginal Productivity as the Basis for Economic Distribution,” (“Om gränsproduktiviteten sansom grundval för den nationalekonomiska fördelningen”), *Ekonomisk Tidskrift*, 2, 305-337 (translated version in K. Wicksell, 1959).
- [33] Wicksteed, P. H. (1894), *Essay on the Coordination of the Laws of Distribution*, (1932 edition), London School of Economics, London, U.K.

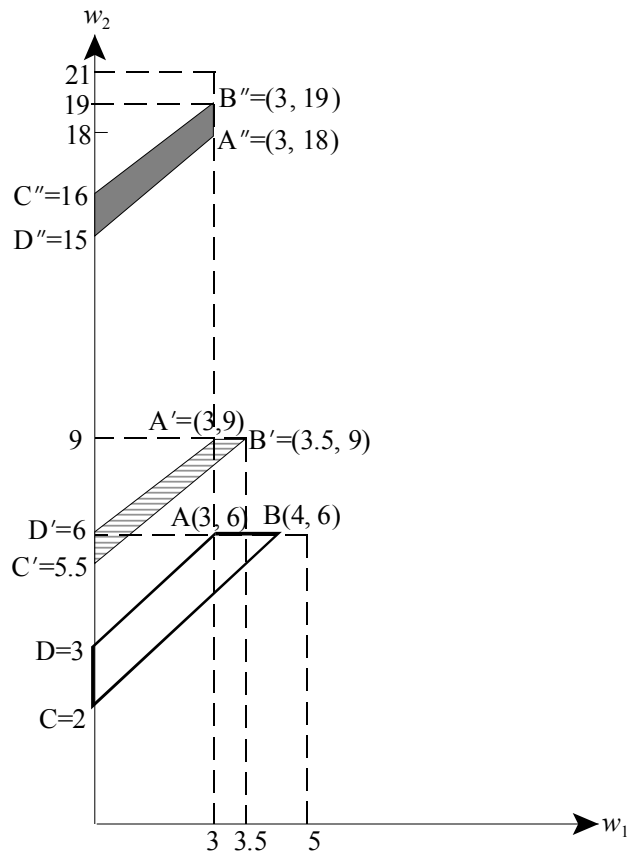


Figure 1: A 2-by-2 Illustrating Example



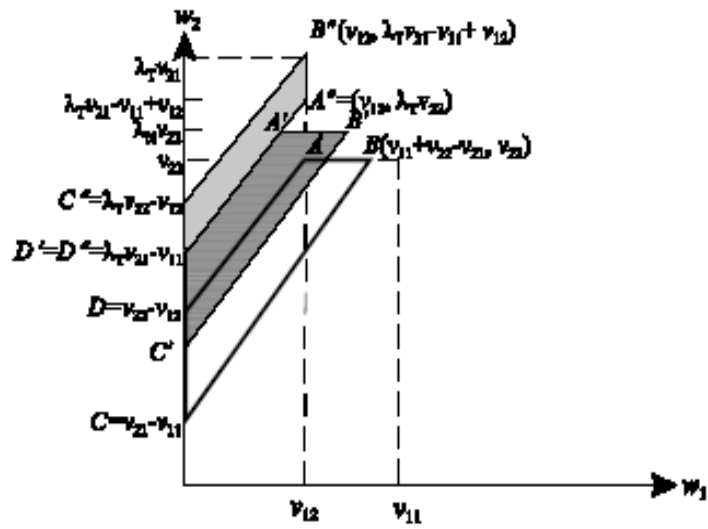


Figure 2: Distribution to Workers

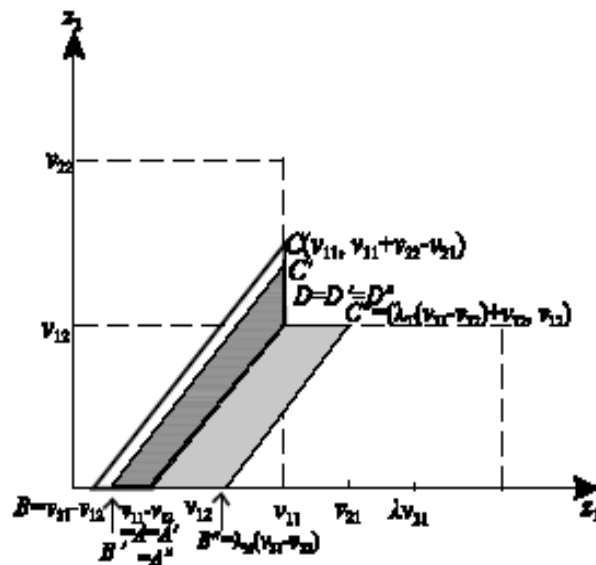


Figure 3: Distribution to Machines