Natural Deduction for Propositional Logic

Bow-Yaw Wang

Institute of Information Science
Academia Sinica, Taiwan

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1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
In our examples, we (informally) infer new sentences.
In natural deduction, we have a collection of proof rules.
  - These proof rules allow us to infer new sentences logically followed from existing ones.
Suppose we have a set of sentences: $\phi_1, \phi_2, \ldots, \phi_n$ (called premises), and another sentence $\psi$ (called a conclusion).
The notation
$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$
is called a sequent.
A sequent is valid if a proof (built by the proof rules) can be found.
We will try to build a proof for our examples. Namely,
$$p \land \neg q \implies r, \neg r, p \vdash q.$$
Suppose we want to prove a conclusion $\phi \land \psi$. What do we do?

- Of course, we need to prove both $\phi$ and $\psi$ so that we can conclude $\phi \land \psi$.

Hence the proof rule for conjunction is

$$
\frac{\phi \quad \psi}{\phi \land \psi} \land i
$$

- Note that premises are shown above the line and the conclusion is below. Also, $\land i$ is the name of the proof rule.
- This proof rule is called “conjunction-introduction” since we introduce a conjunction ($\land$) in the conclusion.
Proof Rules for Natural Deduction – Conjunction

- For each connective, we have introduction proof rule(s) and also elimination proof rule(s).
- Suppose we want to prove a conclusion $\phi$ from the premise $\phi \land \psi$. What do we do?
  - We don’t do any thing since we know $\phi$ already!
- Here are the elimination proof rules:

$$
\frac{\phi \land \psi}{\phi} \quad \land e_1 \\
\frac{\phi \land \psi}{\psi} \quad \land e_2
$$

- The rule $\land e_1$ says: if you have a proof for $\phi \land \psi$, then you have a proof for $\phi$ by applying this proof rule.
- Why do we need two rules?
  - Because we want to manipulate syntax only.
Example

Prove \( p \land q, r \vdash q \land r \).

Proof.

We are looking for a proof of the form:

\[
\begin{array}{cc}
p \land q & r \\
\vdots & \\
q \land r &
\end{array}
\]
Example

Prove $p \land q, r \vdash q \land r$.

Proof.

We are looking for a proof of the form:

\[
\frac{p \land q}{q} \quad \land e_2 \\
\frac{q}{q \land r} \quad \land i
\]

We will write proofs in lines:

1. $p \land q$ premise
2. $r$ premise
3. $q$ $\land e_2$ 1
4. $q \land r$ $\land i$ 3, 2
Suppose we want to prove $\phi$ from a proof for $\neg\neg\phi$. What do we do?

- There is no difference between $\phi$ and $\neg\neg\phi$. The same proof suffices!

Hence we have the following proof rules:

\[
\frac{\phi}{\neg\neg\phi} \quad \neg\neg i \quad \quad \quad \frac{\neg\neg\phi}{\phi} \quad \neg\neg e
\]
Example

Prove $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$.

Proof.

We are looking for a proof like:

\[
\begin{array}{c}
p \\
\neg \neg (q \land r) \\
\vdots \\
\neg \neg p \land r
\end{array}
\]
Example

Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$.

Proof.

We are looking for a proof like:

$$
\begin{align*}
\neg\neg p & \quad \neg\neg i \\
\neg\neg(q \land r) & \quad \neg\neg e \\
q \land r & \quad e_2 \\
\neg\neg p & \quad \neg\neg e_2 \\
r & \quad i \\
\neg\neg p \land r & \quad \land i \\
\neg\neg p \land r & \quad \land e
\end{align*}
$$
Example

Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$.

Proof.

We are looking for a proof like:

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<tbody>
<tr>
<td>1</td>
<td>$p$</td>
<td>premise</td>
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<tr>
<td>2</td>
<td>$\neg\neg(q \land r)$</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>$\neg\neg p$</td>
<td>$\neg\neg i$ 1</td>
</tr>
<tr>
<td>4</td>
<td>$q \land r$</td>
<td>$\neg\neg e$ 2</td>
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<tr>
<td>5</td>
<td>$r$</td>
<td>$\land e_2$ 4</td>
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<tr>
<td>6</td>
<td>$\neg\neg p \land r$</td>
<td>$\land i$ 3, 5</td>
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</table>
Suppose we want to prove $\psi$ from proofs for $\phi$ and $\phi \implies \psi$. What do we do?

- We just put the two proofs for $\phi$ and $\phi \implies \psi$ together.

Here is the proof rule:

$$
\phi \quad \phi \implies \psi \\
\hline
\psi \quad \quad \quad \quad \quad \implies \quad e
$$

This proof rule is also called *modus ponens*.

Here is another proof rule related to implication:

$$
\phi \implies \psi \quad \neg \psi \\
\hline
\neg \phi \quad MT
$$

This proof rule is called *modus tollens*. 
Example

Prove $p \implies (q \implies r), p, \neg r \vdash \neg q$.

Proof.

1. $p \implies (q \implies r)$ premise
2. $p$ premise
3. $\neg r$ premise
4. $q \implies r \implies e \ 2, \ 1$
5. $\neg q \ MT \ 4, \ 3$
Suppose we want to prove $\phi \implies \psi$. What do we do?

- We assume $\phi$ to prove $\psi$. If succeed, we conclude $\phi \implies \psi$ without any assumption.
- Note that $\phi$ is added as an assumption and then removed so that $\phi \implies \psi$ does not depend on $\phi$.

We use “box” to simulate this strategy.

Here is the proof rule:

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\phi \implies \psi \\
\end{array} \quad \implies \quad i
\]

At any point in a box, you can only use a sentence $\phi$ before that point. Moreover, no box enclosing the occurrence of $\phi$ has been closed.
Example

Prove \( \neg q \implies \neg p \vdash p \implies \neg \neg q. \)

Proof.

\[
\begin{array}{c}
\neg q \implies \neg p \\
\hline
\neg p \\
\hline
\neg \neg p \\
\hline
\neg q \\
\hline
\neg \neg q \\
\hline
p \implies \neg \neg q \\
\hline
\end{array}
\]

$\text{1. } \neg q \implies \neg p \quad \text{premise}$

$\text{2. } p \quad \text{assumption}$

$\text{3. } \neg \neg p \quad \neg\neg i \text{ 2}$

$\text{4. } \neg \neg q \quad MT \text{ 1, 3}$

$\text{5. } p \implies \neg \neg q \implies i \text{ 2-4}$
Example

Prove $\vdash p \implies p$.

Proof.

1. $p$ assumption
2. $p \implies p \implies i$ 1 - 1

In the box, we have $\phi \equiv \psi \equiv p$.

Definition

A sentence $\phi$ such that $\vdash \phi$ is called a theorem.
Example

**Prove** \( p \land q \implies r \vdash p \implies (q \implies r) \).

**Proof.**

1. \( p \land q \implies r \) \hspace{1cm} \text{premise}
2. \( p \) \hspace{1cm} \text{assumption}
3. \( q \) \hspace{1cm} \text{assumption}
4. \( p \land q \) \hspace{1cm} \land i 2, 3
5. \( r \) \hspace{1cm} \implies i 4, 1
6. \( q \implies r \) \hspace{1cm} \implies i 3-5
7. \( p \implies (q \implies r) \) \hspace{1cm} \implies i 2-6
Suppose we want to prove $\phi \lor \psi$. What do we do?

- We can either prove $\phi$ or $\psi$.

Here are the proof rules:

- $\phi \lor \psi \lor_i 1$
- $\psi \lor \phi \lor_i 2$

- Note the symmetry with $\land e_1$ and $\land e_2$.

- $\phi \land \psi \land e_1$
- $\psi \land \phi \land e_2$

- Can we have a corresponding symmetric elimination rule for disjunction? Recall

- $\phi \land \psi \land i$
Suppose we want to prove $\chi$ from $\phi \lor \psi$. What do we do?

- We assume $\phi$ to prove $\chi$ and then assume $\psi$ to prove $\chi$.
- If both succeed, $\chi$ is proved from $\phi \lor \psi$ without assuming $\phi$ and $\psi$.

Here is the proof rule:

\[
\begin{array}{c}
\phi \\
\vdots \\
\phi \\
\hline
\phi \lor \psi
\end{array}
\quad
\begin{array}{c}
\psi \\
\vdots \\
\psi \\
\hline
\psi
\end{array}
\quad
\begin{array}{c}
\chi \\
\hline
\chi
\end{array}
\quad
\begin{array}{c}
\chi \\
\hline
\chi
\end{array}
\quad
\begin{array}{c}
\vee e
\end{array}
\]

In addition to nested boxes, we may have parallel boxes in our proofs.
Example

Recall that our syntax does not admit commutativity.

Example

Prove $p \lor q \vdash q \lor p$.

Proof.

$$
\begin{array}{c}
p \lor q \\
\hline
q \lor p
\end{array}
$$

$$
\begin{array}{c}
p \\
\hline
q \lor p
\end{array}
\quad
\begin{array}{c}
q \\
\hline
q \lor p
\end{array}
\quad
\begin{array}{c}
\lor \text{e} 1, 2-3, 4-5
\end{array}
$$

1. $p \lor q$ premise
2. $p$ assumption
3. $q \lor p$ $\lor i_2$ 2
4. $q$ assumption
5. $q \lor p$ $\lor i_1$ 4
6. $q \lor p$ $\lor e$ 1, 2-3, 4-5
Example

Prove \( q \implies r \vdash p \lor q \implies p \lor r \).

Proof.

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<tbody>
<tr>
<td>1</td>
<td>( q \implies r )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( p \lor q )</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>( p )</td>
<td>assumption</td>
</tr>
<tr>
<td>4</td>
<td>( p \lor r )</td>
<td>( \lor i_1 \ 3 )</td>
</tr>
<tr>
<td>5</td>
<td>( q )</td>
<td>assumption</td>
</tr>
<tr>
<td>6</td>
<td>( r )</td>
<td>( \implies e \ 5, \ 1 )</td>
</tr>
<tr>
<td>7</td>
<td>( p \lor r )</td>
<td>( \lor i_2 \ 6 )</td>
</tr>
<tr>
<td>8</td>
<td>( p \lor r )</td>
<td>( \lor e \ 2, \ 3-4, \ 5-7 )</td>
</tr>
<tr>
<td>9</td>
<td>( p \lor q \implies p \lor r )</td>
<td>( \implies i \ 2-8 )</td>
</tr>
</tbody>
</table>
Example

Prove $p \land (q \lor r) \vdash (p \land q) \lor (p \land r)$.

Proof.

1. $p \land (q \lor r)$ premise
2. $p$ $\land e_1$ 1
3. $q \lor r$ $\land e_2$ 1
4. $q$ assumption
5. $p \land q$ $\land i$ 2, 4
6. $(p \land q) \lor (p \land r)$ $\lor i_1$ 5
7. $r$ assumption
8. $p \land r$ $\land i$ 2, 7
9. $(p \land q) \lor (p \land r)$ $\lor i_2$ 8
10. $(p \land q) \lor (p \land r)$ $\lor e$ 3, 4-6, 7-9
Example

Prove \((p \land q) \lor (p \land r) \vdash p \land (q \lor r)\).

Proof.

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<td>premise</td>
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</tr>
<tr>
<td>2</td>
<td>(p \land q)</td>
<td>assumption</td>
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<tr>
<td>3</td>
<td>(p)</td>
<td>(\land e_1) 2</td>
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<tr>
<td>4</td>
<td>(q)</td>
<td>(\land e_2) 2</td>
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<tr>
<td>5</td>
<td>(q \lor r)</td>
<td>(\lor i_1) 4</td>
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<tr>
<td>6</td>
<td>(p \land (q \lor r))</td>
<td>(\land i) 3, 5</td>
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<tr>
<td>7</td>
<td>(p \land r)</td>
<td>assumption</td>
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<td></td>
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<tr>
<td>8</td>
<td>(p)</td>
<td>(\land e_1) 7</td>
<td></td>
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<tr>
<td>9</td>
<td>(r)</td>
<td>(\land e_2) 7</td>
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<td>10</td>
<td>(q \lor r)</td>
<td>(\lor i_2) 9</td>
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<tr>
<td>11</td>
<td>(p \land (q \lor r))</td>
<td>(\land i) 8, 10</td>
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<tr>
<td>12</td>
<td>(p \land (q \lor r))</td>
<td>(\lor e) 1, 2-6, 7-11</td>
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Definition

Contradictions are sentences of the form $\phi \land \neg \phi$ or $\neg \phi \land \phi$.

- Examples:
  - $p \land \neg p$, $\neg (p \lor q \implies r) \land (p \lor q \implies r)$.
- Logically, any sentence can be proved from a contradiction.
  - If $0 = 1$, then $100 \neq 100$.
- Particularly, if $\phi$ and $\psi$ are contradictions, we have $\phi \vdash \psi$.
  - $\phi \vdash \psi$ means $\phi \vdash \psi$ and $\psi \vdash \phi$ (called provably equivalent).
- Since all contradictions are equivalent, we will use the symbol $\bot$ (called “bottom”) for them.
- We are now ready to discuss proof rules for negation.
Since any sentence can be proved from a contradiction, we have

\[ \frac{\bot}{\phi} \quad \bot e \]

When both \( \phi \) and \( \neg \phi \) are proved, we have a contradiction.

\[ \frac{\phi \quad \neg \phi}{\bot} \quad \neg e \]

- The proof rule could be called \( \bot i \). We use \( \neg e \) because it eliminates a negation.
**Example**

Prove $\neg p \lor q \vdash p \implies q$.

**Proof.**

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<tr>
<td>1</td>
<td>$\neg p \lor q$</td>
<td>premise</td>
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<tr>
<td>2</td>
<td>$\neg p$</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>$p$</td>
<td>assumption</td>
</tr>
<tr>
<td>4</td>
<td>$\bot$</td>
<td>$\neg e\ 3,\ 2$</td>
</tr>
<tr>
<td>5</td>
<td>$q$</td>
<td>$\bot e\ 4$</td>
</tr>
<tr>
<td>6</td>
<td>$p \implies q$</td>
<td>$\implies i\ 3-5$</td>
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<tr>
<td>7</td>
<td>$q$</td>
<td>assumption</td>
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<td>8</td>
<td>$p$</td>
<td>assumption</td>
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<td>9</td>
<td>$q$</td>
<td>copy\ 7</td>
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<tr>
<td>10</td>
<td>$p \implies q$</td>
<td>$\implies i\ 8-9$</td>
</tr>
<tr>
<td>11</td>
<td>$p \implies q$</td>
<td>$\lor e\ 1,\ 2-6,\ 7-10$</td>
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</tbody>
</table>
Suppose we want to prove $\neg \phi$. What do we do?

- We assume $\phi$ and try to prove a contradiction. If succeed, we prove $\neg \phi$.

Here is the proof rule:

$$
\phi \\
\vdots \\
\bot \\
\hline 
\neg \phi$$
**Example**

Prove $p \implies q, p \implies \neg q \vdash \neg p$.

**Proof.**

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<tr>
<td>1</td>
<td>$p \implies q$</td>
<td>premise</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$p \implies \neg q$</td>
<td>premise</td>
<td></td>
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<tr>
<td>3</td>
<td>$p$</td>
<td>assumption</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$q$</td>
<td>$\implies e; 3,; 1$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\neg q$</td>
<td>$\implies e; 3,; 2$</td>
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<tr>
<td>6</td>
<td>$\bot$</td>
<td>$\neg e; 4,; 5$</td>
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<tr>
<td>7</td>
<td>$\neg p$</td>
<td>$\neg i; 3-6$</td>
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### Example

Prove $p \land \neg q \implies r, \neg r, p \vdash q$.

#### Proof.

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<tbody>
<tr>
<td>1</td>
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<td>\neg r</td>
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<td>$p$</td>
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<td>\neg q</td>
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<td>$p \land \neg q$</td>
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<td>6</td>
<td>$r$</td>
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<tr>
<td>7</td>
<td>\bot</td>
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<tr>
<td>8</td>
<td>\neg \neg q</td>
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<tr>
<td>9</td>
<td>$q$</td>
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Some rules can actually be derived from others.

Examples

Prove \( p \implies q, \neg q \vdash \neg p \) (modus tollens).

Proof.

1. \( p \implies q \) premise
2. \( \neg q \) premise
3. \( p \) assumption
4. \( q \implies e \ 3, \ 1 \)
5. \( \bot \) \( \neg e \ 4, \ 2 \)
6. \( \neg p \) \( \neg i \ 3-5 \)
Derived Rules

Examples

Prove $p \vdash \neg
\neg p$ ($\neg
\neg i$)

Proof.

1. $p$ premise
2. $\neg p$ assumption
3. $\bot$ $\neg e$ 1, 2
4. $\neg
\neg p$ $\neg i$ 2-3

- These rules can be replaced by their proofs and are not necessary.
  - They are just macros to help us write shorter proofs.
Example

Prove $\neg p \implies \bot \vdash \neg p$ (RAA).

Proof.

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<td>1</td>
<td>$\neg p \implies \bot$</td>
<td>premise</td>
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<tr>
<td>2</td>
<td>$\neg p$</td>
<td>assumption</td>
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<tr>
<td>3</td>
<td>$\bot$</td>
<td>$\implies$</td>
<td>e 2, 1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\neg \neg p$</td>
<td>$\neg i$</td>
<td>2-3</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$p$</td>
<td>$\neg \neg e$</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
Example

Prove $\vdash p \lor \neg p$.

Proof.

1. $\neg (p \lor \neg p)$ assumption
2. $p$ assumption
3. $p \lor \neg p$ $\lor i_1$ 2
4. $\bot$ $\neg e$ 3, 1
5. $\neg p$ $\neg i$ 2-4
6. $p \lor \neg p$ $\lor i_2$ 5
7. $\bot$ $\neg e$ 6, 1
8. $\neg \neg (p \lor \neg p)$ $\neg i$ 1-7
9. $p \lor \neg p$ $\neg \neg e$ 8
Proof Rules for Natural Deduction (Summary)

Conjunction ($\land$)

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad ^i
\]

Disjunction ($\lor$)

\[
\frac{\phi}{\phi \lor \psi} \quad ^i_1 \quad \frac{\psi}{\phi \lor \psi} \quad ^i_2
\]

Implication ($\implies$)

\[
\frac{\phi \quad \psi}{\phi \implies \psi} \quad ^i
\]

\[
\frac{\phi}{\phi \implies \psi} \quad ^i_1 \quad \frac{\psi}{\phi \implies \psi} \quad ^i_2
\]

\[
\frac{\phi \lor \psi \quad \chi}{\chi} \quad ^e
\]

\[
\frac{\phi \quad \phi \implies \psi}{\psi \implies \psi} \quad ^e
\]

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad ^e_1 \quad \frac{\phi \quad \psi}{\phi \land \psi} \quad ^e_2
\]
Proof Rules for Natural Deduction (Summary)

Negation ($\neg$)

\[
\begin{array}{c}
\phi \\
\vdots \\
\bot \\
\hline \\
\neg \phi \\
\end{array}
\quad i
\]

\[
\begin{array}{c}
\phi \\
\hline \\
\bot
\end{array}
\quad e
\]

Contradiction ($\bot$)

(no introduction rule)

\[
\begin{array}{c}
\bot \\
\hline \\
\phi \quad \bot e
\end{array}
\]

Double negation ($\neg \neg$)

(no introduction rule)

\[
\begin{array}{c}
\neg \neg \phi \\
\hline \\
\phi \quad \neg \neg e
\end{array}
\]
Useful Derived Proof Rules

- **MT** (Modus Tollens)
  
  \[
  \frac{\phi \iff \psi \quad \neg \psi}{\neg \phi}
  \]

- **RAA** (Reductio Ad Absurdum)
  
  \[
  \frac{\neg \phi \quad \vdots}{\bot} \quad \frac{\bot}{\phi}
  \]

- **LEM** (Law of Excluded Middle)
  
  \[
  \frac{\phi \quad \neg \phi}{\phi \lor \neg \phi}
  \]
• Recall $p \vdash q$ means $p \models q$ and $q \models p$.

• Here are some provably equivalent sentences:

\[
\begin{align*}
\neg(p \land q) & \vdash \neg q \lor \neg p \\
\neg(p \lor q) & \vdash \neg q \land \neg p \\
p \Rightarrow q & \vdash \neg q \Rightarrow \neg p \\
p \Rightarrow q & \vdash \neg p \lor q \\
p \land q & \Rightarrow p \vdash r \lor \neg r \\
p \land q & \Rightarrow r \vdash p \Rightarrow (q \Rightarrow r)
\end{align*}
\]

• Try to prove them.
Proof by Contradiction

- Although it is very useful, the proof rule RAA is a bit puzzling.

\[
\begin{array}{c}
\neg \phi \\
\vdots \\
\bot \\
\phi \\
\end{array}
\]

RAA

- Instead of proving \( \phi \) directly, the proof rule allows indirect proofs.
  - If \( \neg \phi \) leads to a contradiction, then \( \phi \) must hold.

- Note that indirect proofs are not “constructive.”
  - We do not show why \( \phi \) holds; we only know \( \neg \phi \) is impossible.

- In early 20th century, some logicians and mathematicians chose not to prove indirectly. They are intuitionistic logicians or mathematicians.

- For the same reason, intuitionists also reject

\[
\begin{align*}
\phi \lor \neg \phi & \quad LEM \\
\phi & \quad \neg \neg \phi \neg \neg e
\end{align*}
\]
Proof by Contradiction

**Theorem**

There are $a, b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

**Proof.**

Let $b = \sqrt{2}$. There are two cases:

- If $b^b \in \mathbb{Q}$, we are done since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

- If $b^b \notin \mathbb{Q}$, choose $a = b^b = \sqrt{2}^{\sqrt{2}}$. Then $a^b = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$.

Since $\sqrt{2}^{\sqrt{2}}$, $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, we are done.

An intuitionist would criticize the proof since it does not tell us what $a, b$ give $a^b \in \mathbb{Q}$.

- We know $(a, b)$ is either $(\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$. 
1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
Well-Formedness

Definition

A well-formed formula is constructed by applying the following rules finitely many times:

- **atom**: Every propositional atom \( p, q, r, \ldots \) is a well-formed formula;
- **\( \neg \)**: If \( \phi \) is a well-formed formula, so is \( (\neg \phi) \);
- **\( \land \)**: If \( \phi \) and \( \psi \) are well-formed formulae, so is \( (\phi \land \psi) \);
- **\( \lor \)**: If \( \phi \) and \( \psi \) are well-formed formulae, so is \( (\phi \lor \psi) \);
- **\( \rightarrow \)**: If \( \phi \) and \( \psi \) are well-formed formulae, so is \( (\phi \rightarrow \psi) \).

More compactly, well-formed formulae are defined by the following grammar in Backus Naur form (BNF):

\[
\phi ::= p \mid (\neg \phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)
\]
Inversion Principle

- How do we check if \(((\neg p) \land q) \implies (p \land (q \lor (\neg r)))\) is well-formed?

- Although a well-formed formula needs five grammar rules to construct, the construction process can always be inverted.
  - This is called **inversion principle**.

- To show \(((\neg p) \land q) \implies (p \land (q \lor (\neg r)))\) is well-formed, we need to show both \(((\neg p) \land q)\) and \((p \land (q \lor (\neg r)))\) are well-formed.

- To show \(((\neg p) \land q)\) is well-formed, we need to show both \((\neg p)\) and \(q\) are well-formed.
  - \(q\) is well-formed since it is an atom.

- To show \((\neg p)\) is well-formed, we need to show \(p\) is well-formed.
  - \(p\) is well-formed since it is an atom.

- Similarly, we can show \((p \land (q \lor (\neg r)))\) is well-formed.
The easiest way to decide whether a formula is well-formed is perhaps by drawing its parse tree.
Subformulae

- Given a well-formed formula, its subformulae are the well-formed formulae corresponding to its parse tree.
- For instance, the subformulae of the well-formed formulae $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$ are

  $p$
  $q$
  $r$
  $(\neg p)$
  $(\neg r)$
  $((\neg p) \land q)$
  $(q \lor (\neg r))$
  $(p \land (q \lor (\neg r)))$
  $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$
1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
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   - Completeness of Propositional Logic
We have developed a calculus to determine whether \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid.

- That is, from the premises \( \phi_1, \phi_2, \ldots, \phi_n \), we can conclude \( \psi \).
- Our calculus is syntactic. It depends on the syntactic structures of \( \phi_1, \phi_2, \ldots, \phi_n \), and \( \psi \).

We will introduce another relation between premises \( \phi_1, \phi_2, \ldots, \phi_n \) and a conclusion \( \psi \).

\[
\phi_1, \phi_2, \ldots, \phi_n \models \psi.
\]

- The new relation is defined by ‘truth values’ of atomic formulae and the semantics of logical connectives.
Definition

The set of truth values is \( \{F, T\} \) where \( F \) represents ‘false’ and \( T \) represents ‘true.’

Definition

A valuation or model of a formula \( \phi \) is an assignment from each proposition atom in \( \phi \) to a truth value.
### Definition

Given a valuation of a formula $\phi$, the truth value of $\phi$ is defined inductively by the following truth tables:

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$\phi \land \psi$</th>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$\phi \lor \psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
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<td>T</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$\phi \implies \psi$</th>
<th>$\phi$</th>
<th>$\neg \phi$</th>
<th>$T$</th>
<th>$\bot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
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<td>T</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example

- $\phi \land \psi$ is T when $\phi$ and $\psi$ are T.
- $\phi \lor \psi$ is F when $\phi$ or $\psi$ is T.
- $\bot$ is always F; $\top$ is always T.
- $\phi \Rightarrow \psi$ is T when $\phi$ “implies” $\psi$.

Example

Consider the valuation $\{q \leftrightarrow T, p \leftrightarrow F, r \leftrightarrow F\}$ of $(q \land p) \Rightarrow r$. What is the truth value of $(q \land p) \Rightarrow r$?

Proof.

Since the truth values of $q$ and $p$ are T and F respectively, the truth value of $q \land p$ is F. Moreover, the truth value of $r$ is F. The truth value of $(q \land p) \Rightarrow r$ is T.
Truth Tables for Formulae

- Given a formula $\phi$ with propositional atoms $p_1, p_2, \ldots, p_n$, we can construct a truth table for $\phi$ by listing $2^n$ valuations of $\phi$.

Example

Find the truth table for $(p \implies \neg q) \implies (q \lor \neg p)$.

Proof.

\[
\begin{array}{cccccccc}
 p & q & \neg p & \neg q & p \implies \neg q & q \lor \neg p & (p \implies \neg q) \implies (q \lor \neg p) \\
 F & F & T & T & T & T & T \\
 F & T & T & F & T & T & T \\
 T & F & F & T & T & F & F \\
 T & T & F & F & T & T & T \\
\end{array}
\]
1. Natural Deduction

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Validity of Sequent Revisited

- Informally $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid if we can derive $\psi$ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$.
  - We have formalized “deriving $\psi$ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$” by “constructing a proof in a formal calculus.”

- We can give another interpretation by valuations and truth values.

- Consider a valuation $\nu$ over all propositional atoms in $\phi_1, \phi_2, \ldots, \phi_n, \psi$.
  - By “assumptions $\phi_1, \phi_2, \ldots, \phi_n$,” we mean “$\phi_1, \phi_2, \ldots, \phi_n$ are T under the valuation $\nu$.
  - By “deriving $\psi$,” we mean $\psi$ is also T under the valuation $\nu$.

- Hence, “we can derive $\psi$ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$” actually means “if $\phi_1, \phi_2, \ldots, \phi_n$ are T under a valuation, then $\psi$ must be T under the same valuation.

Semantic Entailment

Definition

We say

\[ \phi_1, \phi_2, \ldots, \phi_n \models \psi \]

holds if for every valuations where \( \phi_1, \phi_2, \ldots, \phi_n \) are T, \( \psi \) is also T. In this case, we also say \( \phi_1, \phi_2, \ldots, \phi_n \) semantically entail \( \psi \).

Examples

- \( p \land q \models p \). For every valuation where \( p \land q \) is T, \( p \) must be T. Hence \( p \land q \models p \).
- \( p \lor q \not\models q \). Consider the valuation \( \{ p \rightarrow T, q \rightarrow F \} \). We have \( p \lor q \) is T but \( q \) is F. Hence \( p \lor q \not\models q \).
- \( \neg p, p \lor q \models q \). Consider any valuation where \( \neg p \) and \( p \lor q \) are T. Since \( \neg p \) is T, \( p \) must be F under the valuation. Since \( p \) is F and \( p \lor q \) is T, \( q \) must be T under the valuation. Hence \( \neg p, p \lor q \models q \).

The validity of \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) is defined by syntactic calculus. \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) is defined by truth tables. Do these two relations coincide?
Soundness Theorem for Propositional Logic

Theorem (Soundness)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and $\psi$ be propositional logic formulae. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds.

Proof.

Consider the assertion $M(k)$:

“For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi (n \geq 0)$ that have a proof of length $k$, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds.”

$k = 1$. The only possible proof is of the form

$1 \quad \phi \quad \text{premise}$

This is the proof of $\phi \vdash \phi$. For every valuation such that $\phi$ is $T$, $\phi$ must be $T$. That is, $\phi \vdash \phi$. 
Proof (cont’d).

Assume $M(i)$ for $i < k$. Consider a proof of the form

1. $\phi_1$ premise
2. $\phi_2$ premise
   
   ... 

n. $\phi_n$ premise
   
   ... 

k. $\psi$ justification

We have the following possible cases for justification:

i. $\land i$. Then $\psi$ is $\psi_1 \land \psi_2$. In order to apply $\land i$, $\psi_1$ and $\psi_2$ must appear in the proof. That is, we have $\phi_1, \phi_2, \ldots, \phi_n \models \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \models \psi_2$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \models \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \models \psi_2$. Hence $\phi_1, \phi_2, \ldots, \phi_n \models \psi_1 \land \psi_2$ (Why?).
Soundness Theorem for Propositional Logic

Proof (cont’d).

ii \( \lor \text{e} \). Recall the proof rule for \( \lor \text{e} \):

\[
\begin{array}{c}
\eta_1 \\
\vdots \\
\psi \\
\eta_2 \\
\vdots \\
\psi \\
\hline
\eta_1 \lor \eta_2 \\
\hline
\psi \\
\hline
\lor \text{e}
\end{array}
\]

In order to apply \( \lor \text{e} \), \( \eta_1 \lor \eta_2 \) must appear in the proof. We have \( \phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2 \). By turning “assumptions” \( \eta_1 \) and \( \eta_2 \) to “premises,” we obtain proofs for \( \phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi \) and \( \phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi \). By inductive hypothesis, \( \phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2 \), \( \phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi \), and \( \phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi \). Consider any valuation such that \( \phi_1, \phi_2, \ldots, \phi_n \) evaluates to T. \( \eta_1 \lor \eta_2 \) must be T. If \( \eta_1 \) is T under the valuation, \( \psi \) is also T (Why?). Similarly for \( \eta_2 \) is T. Thus \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \).
The soundness theorem shows that our calculus does not go wrong. If there is a proof of a sequent, then the conclusion must be true for all valuations where all premises are true.

The theorem also allows us to show the non-existence of proofs.

Given a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, how do we prove there is no proof for the sequent?

- Try to find a valuation where $\phi_1, \phi_2, \ldots, \phi_n$ are T but $\psi$ is F.
1. Natural Deduction

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“φ₁, φ₂, ..., φₙ ⊨ ψ is valid” and “φ₁, φ₂, ..., φₙ ⊨ ψ holds” are very different.

- “φ₁, φ₂, ..., φₙ ⊨ ψ is valid” requires proof search (syntax);
- “φ₁, φ₂, ..., φₙ ⊨ ψ holds” requires a truth table (semantics).

If “φ₁, φ₂, ..., φₙ ⊨ ψ holds” implies “φ₁, φ₂, ..., φₙ ⊨ ψ is valid,”
then our natural deduction proof system is complete.

The natural deduction proof system is both sound and complete. That is
φ₁, φ₂, ..., φₙ ⊨ ψ is valid iff φ₁, φ₂, ..., φₙ ⊨ ψ holds.
We will show the natural deduction proof system is complete.

That is, if $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds, then there is a natural deduction proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

Assume $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$. We proceed in three steps:

1. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi))))$ holds;
2. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi))))$ is valid;
3. $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
Completeness Theorem for Propositional Logic (Step 1)

Lemma

If $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds, then $\models \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ holds.

Proof.

Suppose $\models \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ does not hold. Then there is valuation where $\phi_1, \phi_2, \ldots, \phi_n$ is T but $\psi$ is F. A contradiction to $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.

Definition

Let $\phi$ be a propositional logic formula. We say $\phi$ is a tautology if $\models \phi$.

- A tautology is a propositional logic formula that evaluates to T for all of its valuations.
Our goal is to show the following theorem:

**Theorem**

If $\models \eta$ holds, then $\vdash \eta$ is valid.

Similar to tautologies, we introduce the following definition:

**Definition**

Let $\phi$ be a propositional logic formula. We say $\phi$ is a theorem if $\vdash \phi$.

Two types of theorems:

- If $\vdash \phi$, $\phi$ is a theorem proved by the natural deduction proof system.
- The soundness theorem for propositional logic is another type of theorem proved by mathematical reasoning (less formally).
Proposition

Let $\phi$ be a formula with propositional atoms $p_1, p_2, \ldots, p_n$. Let $l$ be a line in $\phi$’s truth table. For all $1 \leq i \leq n$, let $\hat{p}_i$ be $p_i$ if $p_i$ is $T$ in $l$; otherwise $\hat{p}_i$ is $\neg p_i$. Then

1. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi$ is valid if the entry for $\phi$ at $l$ is $T$;
2. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi$ is valid if the entry for $\phi$ at $l$ is $F$.

Proof.

We prove by induction on the height of the parse tree of $\phi$.

- $\phi$ is a propositional atom $p$. Then $p \vdash p$ or $\neg p \vdash \neg p$ have one-line proof.
- $\phi$ is $\neg \phi_1$.
  - If $\phi$ is $T$ at $l$. Then $\phi_1$ is $F$. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi_1 (\equiv \phi)$.
  - If $\phi$ is $F$ at $l$. Then $\phi_1$ is $T$. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$. Using $\neg \neg i$, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \neg \phi_1 (\equiv \neg \phi)$.
Completeness Theorem for Propositional Logic (Step 2)

Proof (cont’d).

- \( \phi \) is \( \phi_1 \rightarrow \phi_2 \).
  - If \( \phi \) is F at \( l \), then \( \phi_1 \) is T and \( \phi_2 \) is F at \( l \). By IH, \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \) and \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi_2 \). Consider
    
    \[
    \begin{array}{c|c|c}
    1 & \phi_1 \rightarrow \phi_2 & \text{assumption} \\
    \vdots & & \\
    i & \phi_1 & \text{IH} \\
    i + 1 & \phi_2 & \rightarrow e i, 1 \\
    \vdots & & \\
    j & \neg \phi_2 & \text{IH} \\
    j + 1 & \bot & \neg e i+1, j \\
    j + 2 & \neg (\phi_1 \rightarrow \phi_2) & \neg i 1-(j+1)
    \end{array}
    \]
Proof (cont’d).

- \( \phi \) is \( \phi_1 \iff \phi_2 \).
  - If \( \phi \) is \( T \) at \( l \), we have three subcases. Consider the case where \( \phi_1 \) and \( \phi_2 \) are \( F \) at \( l \). Then
    
    \[
    \begin{array}{cccc}
    1 & \phi_1 & \text{assumption} & \\
        & \vdots & & \\
    i & \neg \phi_1 & \text{IH} & \\
    i + 1 & \bot & \neg e\ 1,\ i & \\
    i + 2 & \phi_2 & \bot e\ (i+1) & \\
    i + 3 & \phi_1 \implies \phi_2 & \implies \ i\ 1-(i+2) & \\
    \end{array}
    \]

    The other two subcases are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

Proof (cont’d).

- \( \phi \) is \( \phi_1 \land \phi_2 \).
  - If \( \phi \) is T at \( l \), then \( \phi_1 \) and \( \phi_2 \) are T at \( l \). By IH, we have \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \) and \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_2 \). Using \( \land \) i, we have \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \land \phi_2 \).
  - If \( \phi \) is F at \( l \), there are three subcases. Consider the subcase where \( \phi_1 \) and \( \phi_2 \) are F at \( l \). Then
    
    | i   | \( \neg \phi_1 \)   | IH                          |
    |-----|---------------------|-----------------------------|
    | i + 1 | \( \perp \)         | \( \neg \text{e } 2, i \) |
    | i + 2 | \( \neg(\phi_1 \land \phi_2) \) | \( \neg \text{i } 1-(i+1) \) |

The other two subcases are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

Proof.

- \( \phi \) is \( \phi_1 \lor \phi_2 \).
  - If \( \phi \) is F at \( l \), then \( \phi_1 \) and \( \phi_2 \) are F at \( l \). Then
    
    \[
    \begin{array}{c|c}
    \text{Line} & \text{Step} \\
    \hline
    1 & \phi_1 \lor \phi_2 \text{ assumption} \\
    2 & \phi_1 \text{ assumption} \\
    \vdots & \text{ } \\
    i & \neg \phi_1 \text{ IH} \\
    i + 1 & \bot \text{ } \neg \text{ e 2, i} \\
    i + 2 & \phi_2 \text{ assumption} \\
    \vdots & \text{ } \\
    j & \neg \phi_2 \text{ IH} \\
    j + 1 & \bot \text{ } \neg \text{ e i+2, j} \\
    j + 2 & \bot \text{ } \lor \text{ e 2-(i+1), (i+2)-(j+1)} \\
    j + 3 & \neg (\phi_1 \lor \phi_2) \text{ } \neg \text{ i 1-(j+2)} \\
    \end{array}
    \]
  
  - If \( \phi \) is T at \( l \), there are three subcases. All of them are simple exercises.

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Completeness Theorem for Propositional Logic (Step 2)

**Theorem**

If $\phi$ is a tautology, then $\phi$ is a theorem.

**Proof.**

Let $\phi$ have propositional atoms $p_1, p_2, \ldots, p_n$. Since $\phi$ is a tautology, each line in $\phi$’s truth table is T. By the above proposition, we have the following $2^n$ proofs for $\phi$:

\[
\begin{align*}
\neg p_1, \neg p_2, \ldots, \neg p_n & \vdash \phi \\
p_1, \neg p_2, \ldots, \neg p_n & \vdash \phi \\
\neg p_1, p_2, \ldots, \neg p_n & \vdash \phi \\
\vdots \\
p_1, p_2, \ldots, p_n & \vdash \phi
\end{align*}
\]

We apply the rule LEM and the $\lor$ rule to obtain a proof for $\vdash \phi$. (See the following example.)
Example

Observe that $\vdash p \implies (q \implies p)$. Prove $\vdash p \implies (q \implies p)$.

Proof.

1. $p \lor \neg p$  
   LEM
2. $p$  
   assumption
3. $q \lor \neg q$  
   LEM
4. $q$  
   assumption
   
   \[\vdots\]
   
   i. $p \implies (q \implies p)$  
   $p, q \vdash p \implies (q \implies p)$
   
   i + 1. $\neg q$  
   assumption
   
   \[\vdots\]
   
   j. $p \implies (q \implies p)$  
   $p, \neg q \vdash p \implies (q \implies p)$
   
   j + 1. $p \implies (q \implies p)$  
   $\lor e, 3-i, (i+1)-j$
   
   j + 2. $\neg p$  
   assumption
   
   j + 3. $q \lor \neg q$  
   LEM
   
   j + 4. $q$  
   assumption
   
   \[\vdots\]
   
   k. $p \implies (q \implies p)$  
   $\neg p, q \vdash p \implies (q \implies p)$
   
   k + 1. $\neg q$  
   assumption
   
   \[\vdots\]
   
   l. $p \implies (q \implies p)$  
   $\neg p, \neg q \vdash p \implies (q \implies p)$
   
   l + 1. $p \implies (q \implies p)$  
   $\lor e, (j+3), (j+4)-k, (k+1)-l$
   
   l + 2. $p \implies (q \implies p)$  
   $\lor e, 1, 2-(j+1), (j+2)-(l+1)$
Completeness Theorem for Propositional Logic (Step 3)

**Lemma**

If \( \phi_1 \implies (\phi_2 \implies (\cdots (\phi_n \implies \psi)\cdots)) \) is a theorem, then \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid.

**Proof.**

Consider

<p>| | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( \phi_1 )</td>
<td>premise</td>
</tr>
<tr>
<td>(2)</td>
<td>( \phi_2 )</td>
<td>premise</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(n)</td>
<td>( \phi_n )</td>
<td>premise</td>
</tr>
<tr>
<td>(i)</td>
<td>( \phi_1 \implies (\phi_2 \implies (\cdots (\phi_n \implies \psi)\cdots)) )</td>
<td>theorem</td>
</tr>
<tr>
<td>(i+1)</td>
<td>( \phi_2 \implies (\cdots (\phi_n \implies \psi)\cdots) )</td>
<td>(\implies ) e 1, (i)</td>
</tr>
<tr>
<td>(i+2)</td>
<td>( \phi_3 \implies (\cdots (\phi_n \implies \psi)\cdots) )</td>
<td>(\implies ) e 2, ((i+1))</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(i+n-1)</td>
<td>( \phi_n \implies \psi )</td>
<td>(\implies ) e ((n-1)), ((i+n-2))</td>
</tr>
<tr>
<td>(i+n)</td>
<td>( \psi )</td>
<td>(\implies ) e (n), ((i+n-1))</td>
</tr>
</tbody>
</table>